

Magnetized Turbulent Dynamo in Protogalaxies

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ABSTRACT

The prevailing theory for the origin of cosmic magnetic fields is that they have been amplified to their present values by the turbulent dynamo inductive action in the protogalactic and galactic medium. Up to now, in calculation of the turbulent dynamo, it has been customary to assume that there is no back reaction of the magnetic field on the turbulence, as long as the magnetic energy is less than the turbulent kinetic energy. This assumption leads to the kinematic dynamo theory. However, the applicability of this theory to protogalaxies is rather limited. The reason is that in protogalaxies the temperature is very high, and the viscosity is dominated by magnetized ions. As the magnetic field strength grows in time, the ion cyclotron time becomes shorter than the ion collision time, and the plasma becomes strongly magnetized. As a result, the ion viscosity becomes the Braginskii viscosity. Thus, in protogalaxies the back reaction sets in much earlier, at field strengths much lower than those which correspond to field-turbulence energy equipartition, and the turbulent dynamo becomes what we call the magnetized turbulent dynamo. In this paper we lay the theoretical groundwork for the magnetized turbulent dynamo. In particular, we predict that the magnetic energy growth rate in the magnetized dynamo theory is up to ten times larger than that in the kinematic dynamo theory. We also briefly discuss how the Braginskii viscosity can aid the development of the inverse cascade of magnetic energy after the energy equipartition is reached.

Subject headings: galaxies: magnetic fields — MHD — turbulence — methods: analytical

1. Introduction

One of the most important and challenging questions in astrophysics is the origin of strong and large-scale magnetic fields in galaxies and protogalaxies. It is now widely accepted that the strong cosmic magnetic fields were produced by the turbulent dynamo inductive action driven by the fluid motions in a galactic and/or protogalactic medium (Vainshtein & Zel’dovich 1972; Zweibel & Heiles 1997; Kulsrud 2000). Yet full understanding of all stages of this production has not been achieved. There are two alternative theories on how and when the magnetic fields have been produced. The first theory, the galactic dynamo theory, also known as the α – Ω dynamo theory, states that the fields have been primarily amplified in differentially rotating galactic disks after the galaxies had been formed (Parker 1971; Vainshtein & Ruzmaikin 1972; Beck *et al.* 1996). The galactic dynamo involves several crucial unsolved problems (Rosner & Deluca 1989; Zweibel & Heiles 1997, Kulsrud 1999). The main problem is that in the α – Ω theory it seems to be extremely difficult to expel a fraction of the magnetic flux from a galactic disk in order to produce the net magnetic flux (Rafikov & Kulsrud 2000). In addition, observations indicate the presence of microgauss magnetic fields in galaxy clusters and in early galaxies at high redshifts (Perry 1994; Kronberg 1994). It is hard to explain such strong fields by the galactic dynamo theory (Zweibel & Heiles 1997). In this paper we accept the second theory for the origin of cosmic magnetic fields, *the primordial dynamo theory*, which states that the galactic and extragalactic magnetic fields have primarily been produced in protogalaxies, i. e. before the galaxies were formed (Pudritz & Silk 1989; Kulsrud & Anderson 1992; Kulsrud *et al.* 1997; Kulsrud 2000). Of course, these fields were subsequently modified in the rotating galactic disks after the galaxies were formed.

In order to understand how the magnetic fields can be built up in protogalaxies, let us briefly discuss the physical conditions that were present there ¹. Let us assume the following typical values for the total mass M of a protogalaxy, the total to baryon mass ratio ξ , and the protogalaxy size L : $M \sim 10^{12} M_\odot$, $\xi \sim 10$, and $L \sim 0.2$ Mpc. Then, the number density of the gas in the protogalaxy is $n \sim \xi^{-1} M L^{-3} m_p^{-1} \sim 5 \times 10^{-4} \text{ cm}^{-3} \propto \xi^{-1} M L^{-3}$ (m_p is the proton mass, for convenience, in this paragraph we give the scaling of physical parameters with the three “basic” parameters ξ , M and L). Assuming the energy virial equilibrium in the protogalaxy, we easily estimate the gas temperature, $T \sim k_B^{-1} G m_p M L^{-1} \sim 2 \times 10^6 \text{ K} \propto M L^{-1}$ (G and k_B are the gravitational and the Boltzmann constants). This temperature is very high, while the density is very low. As a result, the gas is fully ionized, and the viscosity is dominated by ions, not by neutrals. The ion collision time is $t_i \sim 20 T^{3/2} n^{-1} \Lambda_c^{-1} \text{ sec} \sim 3 \times 10^{12} \text{ sec} \propto \xi M^{1/2} L^{3/2}$ (in this formula the temperature is in degrees K, the density is in cm^{-3} , and $\Lambda_c \sim 30$ is the Coulomb logarithm assumed to be independent of ξ , M and L , Braginskii 1965). The virial thermal speed is $V_T \sim (2 k_B T / m_p)^{1/2} \sim 2 \times 10^7 \text{ cm/s} \propto M^{1/2} L^{-1/2}$. The ion kinematic viscosity can be estimated as $\nu = 0.96 k_B T t_i / m_p \sim 5 \times 10^{26} \text{ cm}^2/\text{s} \propto \xi M^{3/2} L^{1/2}$ (Braginskii 1965). The Spitzer resistivity is $\eta_s = 6.53 \times 10^{12} T^{-3/2} \Lambda_c \text{ cm}^2/\text{s} \sim 8 \times 10^4 \text{ cm}^2/\text{s} \propto M^{-3/2} L^{3/2}$ (in this formula the temperature is

¹The numerical quantities given below refer to common values at the time (redshift) when the protogalaxies form.

in degrees K, and the Coulomb logarithm is assumed to be constant, $\Lambda_c \sim 30$, Spitzer 1962). Now we estimate the Reynolds and the Prandtl numbers, $R \sim V_T/k_0\nu \sim 10^4 \propto \xi^{-1}M^{-1}$ ($k_0 = 2\pi/L$ is the minimal wave number in the protogalaxy) and $Pr \sim \nu/\eta_s \sim 10^{22} \propto \xi M^3 L^{-1}$, they are very large. The viscous cutoff scale of the turbulence can be estimated as $2\pi k_\nu^{-1} \sim R^{-3/4}L \sim 10^{-3}L \propto \xi^{3/4}M^{3/4}L$, the resistive cutoff scale for the magnetic field as $2\pi k_{\eta_s}^{-1} \sim Pr^{-1/2}R^{-3/4}L \sim 10^{-14}L \propto \xi^{1/4}M^{-3/4}L^{3/2}$, and the ion mean free path as $\lambda_i \sim R^{-1}L \sim 10^{-4}L \propto \xi ML$. Thus, there is a hierarchy of scales in the protogalaxy, $L \gg 2\pi k_\nu^{-1} \gg \lambda_i \gg 2\pi k_{\eta_s}^{-1}$. Therefore, we can use the single-fluid magnetohydrodynamic (MHD) equations for the description of plasma, and can consider the plasma to be nonresistive and incompressible on scales $L > 2\pi k^{-1} \gtrsim \lambda_i$.²

It is very important that in a protogalaxy the ion cyclotron period in the magnetic field, $\omega_i^{-1} = m_p c / eB \sim 10^{-4} B^{-1} \text{ G} \cdot \text{sec}$, is shorter than the ion collision time, $t_i \sim 3 \times 10^{12} \text{ sec}$, provided that the field strength is larger than $\sim 10^{-17} - 10^{-16} \text{ G}$. On the other hand, the magnetic energy becomes comparable to the kinetic energy of the smallest turbulent eddies (which are on the viscous cutoff scale) if the field strength exceeds $\sim (4\pi m_p n V_T^2 R^{-1/2})^{1/2} \sim 10^{-7} \text{ G}$. Thus, there is a very broad range of magnetic field strengths, at which the magnetic pressure and tension are still negligible, while the presence of the field is already important. This is because the plasma is strongly magnetized, $\omega_i^{-1} \ll t_i$, and the magnetic field controls the microscopic motions of ions, so that the viscous forces are given by the Braginskii viscous stress [Braginskii 1965; see also eq. (1)] and are different from the viscous forces in a field-free plasma.

²Note that the Debye length $(k_B T / 2\pi n e^2)^{1/2} \sim 6 \times 10^5 \text{ cm} \propto \xi^{1/2} L$ is extremely small in protogalaxies. Also we can consider the plasma to be incompressible on scales $< L$ because the plasma velocities at the largest scales, $\sim L$, are of the order of the sound speed, and all velocities at smaller scales are smaller. For estimation purposes, we can use the MHD equations even for scales $2\pi k^{-1} \ll \lambda_i$, if we reduce the molecular viscosity by factor $(k\lambda_i)^{-1} \ll 1$.

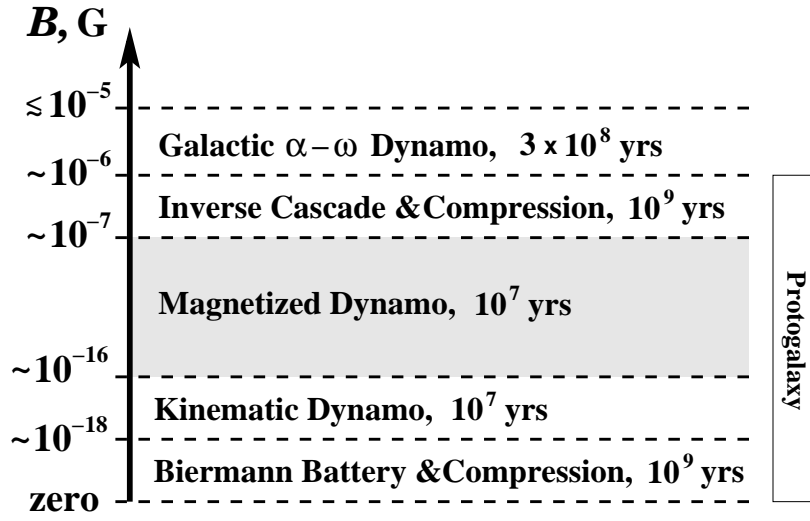


Fig. 1.— There are five major stages of the production of galactic and extragalactic magnetic fields, as the field strength grows up from zero to $\lesssim 10^{-5}$ gauss. In this paper we consider the magnetized turbulent dynamo stage in protogalaxies, shaded in the plot.

We believe that there are five major stages of the production of the strong cosmic magnetic fields, see Figure 1. During the first stage, in a protogalaxy undergoing gravitational collapse, the Biermann battery builds a seed magnetic field linearly in time, on a time scale approximately equal to the free-fall time, ~ 1 billion years. The resulting seed field is of the order of $\sim 10^{-18}$ G on the viscous scale (Pudritz & Silk 1989; Kulsrud *et al.* 1997; Davies & Widrow 2000).

During the second stage, when the plasma is unmagnetized, $\omega_i^{-1} \gg t_i$, the seed field is exponentially amplified by the kinematic turbulent dynamo inductive action. The kinematic dynamo builds the field up to approximately 10^{-16} G, when the plasma becomes magnetized, $\omega_i^{-1} \ll t_i$. The time scale of the kinematic dynamo is very short, it is of the order of the smallest eddy turnover time (Kulsrud *et al.* 1997), ~ 10 million years, which is much smaller than the gravitational collapse time of the protogalaxy, several billion years. The main assumption of the kinematic dynamo theory and, basically, the strict definition of it, are that the growing magnetic field stays so weak, that it does not affect the fluid motions, i. e. that there is no *the back reaction* of the field on the turbulence. The results we obtain in this paper reduce to those of the kinematic dynamo theory in the limit of very weak field strengths (i. e. when the plasma is unmagnetized, and there is no the back reaction).

The third stage starts when the field grows above $\sim 10^{-16}$ G, the ion cyclotron time in the magnetic field becomes shorter than the ion collision time, and the plasma becomes strongly magnetized. During this stage the magnetic field strongly affects the dynamics of the turbulent motions on the viscous scales ³, by completely changing the viscosity (see Section 2), despite the fact that the magnetic energy is still very small compared to the kinetic energy of the turbulence! We call this stage as *the magnetized turbulent dynamo*. In previous theories this stage has not been recognized. The main goal of this paper is to construct a theoretical model for it. The time scale of the magnetized dynamo is about the same as the kinematic dynamo time scale, ~ 10 million years.

So far the field scale is of the order of the viscous scale or less, and the magnetic field is incoherent in space. The fourth stage starts when the magnetic field strength grows above $\sim 10^{-7}$ G. The field energy becomes comparable to the kinetic energy of the smallest turbulent eddies, and the Lorentz forces become dynamically important in the plasma. During this stage the turbulent motions are dissipated by the growing field, the turbulent energy spectrum becomes truncated at larger and larger scales, and the turbulent energy is eventually transferred into a large-scale strong magnetic field with its energy comparable to the kinetic energy of the fluid motions on the largest scales in the protogalaxy (which is approximately the same as the thermal energy). This process (which is still under debate) is called the inverse cascade (Vainshtein 1982; Kulsrud & Anderson 1992; Beck *et al.* 1996; Kulsrud 2000). We discuss it qualitatively in Section 5. The

³The turbulent motions on the viscous scales are the most important in the dynamo theory. This is because the magnetic field is primarily amplified by the turbulent eddies on these scales. These eddies are the smallest ones, they have the shortest turnover times and produce the largest velocity shearing.

time scale available for the inverse cascade process is of the order of the largest eddy turnover time, ~ 1 billion years.

The turbulent dynamos and the inverse cascade may not have time to amplify the field up to microgauss values, which are observed in galaxies. A crucial question is how far they go? This paper addresses this question and is concerned with the rate of the magnetic field built up by the magnetized turbulent dynamo.

Finally, the fifth stage is the galactic dynamo, which happens in the differentially rotating galactic disc after the galaxy is formed. This process modifies the strong field that was initially built up in the protogalaxy, on a time scale of the order of the rotation time of the galaxy, ~ 300 million years. The galactic dynamo theory is not discussed in this paper.

In Section 2 we formulate the basic equations of the magnetized dynamo theory. In Section 3 we calculate the statistical correlation functions for turbulent velocities, \mathbf{V} , in a strongly magnetized plasma by making use of the quasilinear expansion in time of the MHD equations for both the velocities and the magnetic field, similar to the expansion used by Kulsrud and Anderson (1992). We find that, contrary to the Kolmogorov velocities, the turbulent velocities in the magnetized plasma are strongly anisotropic on the viscous scale, as one might expect, because the magnetic field sets “a preferred axis in space”. In our calculations we make two working hypotheses. First, we assume that for the purpose of magnetic energy calculation, the tensor $b_{\alpha\beta} = \hat{b}_\alpha \hat{b}_\beta$, where $\hat{\mathbf{b}}$ is the field unit vector, can be taken to be constant in space in the beginning of the expansion in time. This is our first hypothesis, which basically relies on our assumption that in the magnetized turbulent dynamo case the magnetic field has a folding structure similar to the one that exists in the kinematic turbulent dynamo case (see Figure 2; Maron & Cowley 2001; Schekochihin *et al.* 2002). Second, we find that there are velocity modes which are not damped by the Braginskii viscous forces and, therefore, are divergent unless we incorporate the MHD non-linear inertial terms into our quasilinear expansion. We assume that these non-linear terms, which limit the divergent velocity modes by coupling them to other modes, may be included into our theory by allowing for rotation of velocity vectors relative to the magnetic field unit vectors ⁴. This is our second hypothesis. In Section 4 we use the correlation functions for the turbulent velocities found in Section 3 to calculate the evolution of the magnetic energy in the magnetized turbulent plasma. We start with calculations of the total magnetic energy growth rate in Subsection 4.1. In Subsection 4.2 we derive the integro-differential mode coupling equation for the magnetic energy spectrum. Our mode coupling equation is a more general version of the corresponding equation of Kulsrud and Anderson (1992), obtained for the kinematic dynamo. In Subsection 4.3 we consider the magnetic energy spectrum on small subviscous scales. On these scales the mode coupling equation greatly simplifies and becomes a homogeneous differential equation. Finally, in Section 5 we give our conclusions. We also discuss the peculiarities of the inverse cascade in a strongly magnetized turbulent plasma.

⁴This rotation is essentially due to Coriolis forces that make the velocity rotate differently than the field direction.

2. Basic Magnetized Dynamo Equations

Hereafter we consider the magnetized turbulent dynamo stage, the shaded region in Figure 1. During this stage the plasma is strongly magnetized, $\omega_i t_i \gg 1$, but the magnetic energy is still small compared to the turbulent kinetic energy, and therefore, the magnetic Lorentz forces can be neglected. The viscous forces acting on turbulent velocities \mathbf{V} in a strongly magnetized incompressible fully ionized plasma are determined by the Braginskii viscosity stress tensor (Braginskii 1965),

$$\pi_{\alpha\beta} = -\nu(3\hat{b}_\alpha\hat{b}_\beta - \delta_{\alpha\beta})\hat{b}_\mu\hat{b}_\nu V_{\mu,\nu}, \quad (1)$$

where $\hat{\mathbf{b}} = \mathbf{B}/B$ is the unit vector along the magnetic field. Note, that this stress tensor depends on the field unit vector, but is independent of the magnetic strength (as long as the plasma is strongly magnetized, and $\omega_i t_i \gg 1$). Thus, during the magnetized turbulent dynamo stage the magnetic field strongly affects the turbulent motions on the viscous scales by changing the viscous forces, even though the Lorentz forces are negligible.

The MHD equations for the turbulent velocities \mathbf{V} in an incompressible strongly magnetized plasma, neglecting the Lorentz forces, are (Landau & Lifshitz 1984)

$$\begin{aligned} \partial_t V_\alpha &= -P'_{,\alpha} + f_\alpha - \pi_{\alpha\beta,\beta} - (V_\alpha V_\beta)_{,\beta} \\ &= -P''_{,\alpha} + f_\alpha + 3\nu(\hat{b}_\alpha\hat{b}_\beta\hat{b}_\mu\hat{b}_\nu V_{\mu,\nu})_{,\beta} - (V_\alpha V_\beta)_{,\beta}, \end{aligned} \quad (2)$$

$$V_{\alpha,\alpha} = 0, \quad (3)$$

where \mathbf{f} is the force driving the turbulence, and P' is the hydrodynamic pressure. Here and below we always assume summation over repeated indices. In order to shorten notations, we use $\partial_t \stackrel{\text{def}}{=} \partial/\partial t$, and spatial derivatives are assumed to be taken with respect to all indices that are listed after “,” signs⁵. To obtain the second line of equation (2), we use formula (1) for the viscous stress and incorporate the isotropic part of the stress into the pressure P'' .

It is difficult to solve equations (2) and (3) directly because they are very complicated. We also do not know the exact expression of the driving force \mathbf{f} , but its statistics is the same as for an unmagnetized plasma. Therefore, let us proceed as follows. We represent \mathbf{f} by introducing subsidiary incompressible turbulent velocities \mathbf{U} which, by definition, satisfy equations

$$\partial_t U_\alpha = -P'''_{,\alpha} + f_\alpha + (1/5)\nu\Delta U_\alpha - (U_\alpha U_\beta)_{,\beta}, \quad (4)$$

$$U_{\alpha,\alpha} = 0, \quad (5)$$

where $\Delta U_\alpha = U_{\alpha,\beta\beta}$. (\mathbf{U} will be essentially the Kolmogorov turbulent velocities.)

Let us analyze and compare equations (2) and (4). First, if for the moment we formally average the Braginskii viscosity term $3\nu(\hat{b}_\alpha\hat{b}_\beta\hat{b}_\mu\hat{b}_\nu V_{\mu,\nu})_{,\beta}$ in equation (2) over all directions of an isotropic

⁵For example, $(V_\alpha V_\beta)_{,\gamma} \equiv V_\beta(\partial V_\alpha/\partial x_\gamma) + V_\alpha(\partial V_\beta/\partial x_\gamma)$, and $V_{\alpha,\beta\gamma} \equiv \partial^2 V_\alpha/\partial x_\beta \partial x_\gamma$.

magnetic field, then it reduces to $(1/5)\nu\Delta V_\alpha$, which coincides with the isotropic viscosity term in equation (4). Therefore, $\nu_{\text{eff}} = (1/5)\nu$ could be considered as an *effective reduced viscosity* for an incompressible fully ionized plasma in the presence of a magnetic field that is isotropically tangled on subviscous scales. In other words, the Braginskii viscous forces “are doing worse” at dissipating the turbulent motions, as compared with the standard isotropic viscous forces in a field-free plasma. Second, note that equations (2) and (4) have the same driving force \mathbf{f} . By taking the driving force to be the same, we assume that this force comes from larger turbulent eddies. These larger eddies are on scales larger than the viscous scales, and therefore, these eddies “do not know” whether the viscous forces are of the Braginskii type or of the standard isotropic type.

Now, note that equation (4) is a familiar hydrodynamic equation with a standard isotropic viscosity term. However, it has a reduced molecular viscosity, $(1/5)\nu$ instead of ν . Therefore, we assume that the solution of equations (4) and (5) is the incompressible, homogeneous, isotropic and stationary Kolmogorov turbulence with the effective reduced viscosity

$$\nu_{\text{eff}} = (1/5)\nu. \quad (6)$$

As a result, the statistics of the Fourier coefficients of the turbulent velocities \mathbf{U} ,

$$\tilde{U}_{\mathbf{k}\alpha}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{U}_{\mathbf{k}\alpha}(t) e^{i\omega t} dt, \quad \tilde{U}_{\mathbf{k}\alpha}(t) = \frac{1}{L^3} \int_{-L/2}^{L/2} U_\alpha(t, \mathbf{r}) e^{-i\mathbf{k}\mathbf{r}} d^3\mathbf{r}, \quad (7)$$

is given by the following formulas ⁶:

$$\langle \tilde{U}_{\mathbf{k}\alpha}(\omega) \rangle = 0, \quad (8)$$

$$\langle \tilde{U}_{\mathbf{k}\alpha}(\omega) \tilde{U}_{\mathbf{k}'\beta}(\omega') \rangle = \langle \tilde{U}_{-\mathbf{k}\alpha}^*(-\omega) \tilde{U}_{\mathbf{k}'\beta}(\omega') \rangle = J_{\omega k} \delta_{\alpha\beta}^\perp \delta_{\mathbf{k}', -\mathbf{k}} \delta(\omega' + \omega), \quad (9)$$

$$\delta_{\alpha\beta}^\perp \stackrel{\text{def}}{=} \delta_{\alpha\beta} - \hat{k}_\alpha \hat{k}_\beta, \quad (10)$$

$$J_{\omega k} = J_{0k} (1 + \tau^2 \omega^2)^{-1}, \quad (11)$$

$$\tau(k) = \tau(0) (k/k_0)^{-2/3} = (1/k_0 U_0) (k/k_0)^{-2/3}, \quad (12)$$

$$J_{0k} \approx \begin{cases} (U_0/6\pi k_0)(k/k_0)^{-13/3}, & k_0 \leq k \leq k_\nu = (5U_0/k_0\nu)^{3/4} k_0, \\ 0, & k < k_0 = 2\pi/L, \quad k > k_\nu. \end{cases} \quad (13)$$

Here and below $\langle \dots \rangle$ means ensemble average over all realizations of the turbulence, $\delta(\omega' + \omega)$ is the Dirac δ -function, $\delta_{\alpha\beta}$ and $\delta_{\mathbf{k}', -\mathbf{k}}$ are the one- and three-dimensional Kronecker symbols respectively, $\hat{\mathbf{k}} = \mathbf{k}/k$ is the unit vector along k , $U_0 \sim V_T$ is the largest eddy velocity, $k_0 = 2\pi/L$ is the smallest wave number of the turbulence, and $k_\nu \sim (U_0/k_0\nu_{\text{eff}})^{3/4} k_0 = (5U_0/k_0\nu)^{3/4} k_0$ is the viscous cutoff wave number of the turbulence ⁷. In equation (9) we keep only the normal part of

⁶Note that these formulas are similar to those of Kulsrud *et al.* 1997, but differ by numerical coefficients from those of Kulsrud & Anderson 1992.

⁷ k_ν is determined by the balance between the inertial and the viscous terms of eq. (4), $1/\tau(k_\nu) \sim \nu_{\text{eff}} k_\nu^2$.

the turbulence and drop the helical part, since the latter is negligible on the scales of the smallest turbulent eddies, which are the principal drivers of the field evolution (Kulsrud & Anderson 1992). To obtain equation (11), we assume that the time correlation function of the turbulent velocities has an exponential profile ⁸,

$$\langle \tilde{U}_{\mathbf{k}\alpha}(t) \tilde{U}_{\mathbf{k}'\beta}(t') \rangle = \frac{J_{0k}}{2\tau} e^{-|t-t'|/\tau} \delta_{\alpha\beta}^{\perp} \delta_{\mathbf{k}', -\mathbf{k}}, \quad (14)$$

where τ is the eddy decorrelation time given by equation (12) for the Kolmogorov turbulence. Note that the averaged total kinetic energy of the fluid motions, per unit mass, is

$$\frac{1}{2} \langle [\mathbf{U}(t, \mathbf{r})]^2 \rangle = \frac{1}{2} \sum_{\mathbf{k}} \langle |\tilde{\mathbf{U}}_{\mathbf{k}}(t)|^2 \rangle = \frac{1}{2} \sum_{\mathbf{k}} \frac{J_{0k}}{\tau(k)} = \frac{1}{2} \int_{k_0}^{k_\nu} I(k) dk = \frac{1}{2} U_0^2, \quad (15)$$

where the Kolmogorov energy spectrum is $I(k) = 4\pi k^2 (L/2\pi)^3 J_{0k}/\tau = (2/3)(U_0^2/k_0)(k/k_0)^{-5/3}$ if $k \in [k_0, k_\nu]$, as it should be (Kulsrud *et al.* 1997).

Now let us subtract equations (4) and (5) from equations (2) and (3) to eliminate the unknown driving force \mathbf{f} , and let us introduce *the back-reaction velocity* $\mathbf{v} \stackrel{\text{def}}{=} \mathbf{V} - \mathbf{U}$. We have

$$\partial_t v_\alpha = -P_{,\alpha} + 3\nu(b_{\alpha\beta\mu\nu}v_{\mu,\nu} + b_{\alpha\beta\mu\nu}U_{\mu,\nu})_{,\beta} - (1/5)\nu\Delta U_\alpha - (v_\alpha U_\beta + U_\alpha v_\beta + v_\alpha v_\beta)_{,\beta}, \quad (16)$$

$$v_{\alpha,\alpha} = 0, \quad (17)$$

where the pressure $P = P'' - P'''$. Here and below we use the following symmetric tensors:

$$b_{\alpha\beta\gamma\delta} \stackrel{\text{def}}{=} \hat{b}_\alpha \hat{b}_\beta \hat{b}_\gamma \hat{b}_\delta, \quad b_{\alpha\beta\gamma} \stackrel{\text{def}}{=} \hat{b}_\alpha \hat{b}_\beta \hat{b}_\gamma, \quad b_{\alpha\beta} \stackrel{\text{def}}{=} \hat{b}_\alpha \hat{b}_\beta. \quad (18)$$

Velocity \mathbf{v} , which satisfies equations (16) and (17), can be considered as the correction to the Kolmogorov velocity \mathbf{U} . This correction is non-zero only on the viscous scales, and results from the strong back reaction of the field on the turbulence via the Braginskii viscosity tensor (1).

Finally, the MHD equation for the magnetic field \mathbf{B} is (Landau & Lifshitz 1984)

$$\partial_t B_\alpha = V_{\alpha,\beta} B_\beta - V_\beta B_{\alpha,\beta}, \quad (19)$$

where the plasma velocities $\mathbf{V} = \mathbf{U} + \mathbf{v}$ are incompressible, and we neglect resistivity. Consequently, the equations for the magnetic field squared, B^2 , and for the magnetic field unit vector, $\hat{\mathbf{b}}$, are

$$\partial_t B^2 = 2V_{\alpha,\beta} B_\alpha B_\beta - V_\beta (B^2)_{,\beta}, \quad (20)$$

$$\partial_t \hat{b}_\alpha = V_{\alpha,\beta} \hat{b}_\beta - V_{\beta,\gamma} b_{\alpha\beta\gamma} - V_\beta \hat{b}_{\alpha,\beta}. \quad (21)$$

⁸Using a Gaussian time correlation profile, $\langle \mathbf{U}(t)\mathbf{U}(t') \rangle \propto e^{-(t-t')^2/2\tau^2}$, would be more appropriate. In this case equation (11) would become $J_{\omega k} = J_{0k} e^{-\tau^2 \omega^2/2}$. However, we prefer the exponential profile because it is easier to deal with (e.g., for Gaussian integrals it is not possible to close integration contours at infinity in the complex plane).

3. Statistics of Turbulent Velocities in Strongly Magnetized Plasmas

In order to find the evolution of the magnetic field \mathbf{B} , we must derive the correlation functions for the total velocities \mathbf{V} , resulting from the Braginskii viscosity. We calculate these velocity correlation functions in this section.

Let us assume that we know the magnetic field at zero time, $\mathbf{B}|_{t=0} = {}^0\mathbf{B}(\mathbf{r})$ and $\hat{\mathbf{b}}|_{t=0} = {}^0\hat{\mathbf{b}}(\mathbf{r})$, and that the back-reaction velocity \mathbf{v} is initially zero, $\mathbf{v}|_{t=0} = {}^0\mathbf{v}(\mathbf{r}) \equiv 0$.⁹ Then we advance the magnetic field and the back-reaction velocity to some future time, $t > 0$, by the nonlinear terms, i. e. by integrating equations (16), (19) and (21) twice in time. This quasi-linear expansion procedure is similar to the calculations of Kulsrud and Anderson (1992). Considering t as the expansion parameter¹⁰, up to the second order, we have

$$\mathbf{B}(t, \mathbf{r}) = {}^0\mathbf{B}(\mathbf{r}) + {}^1\mathbf{B}(t, \mathbf{r}) + {}^2\mathbf{B}(t, \mathbf{r}), \quad (22)$$

$$\hat{\mathbf{b}}(t, \mathbf{r}) = {}^0\hat{\mathbf{b}}(\mathbf{r}) + {}^1\hat{\mathbf{b}}(t, \mathbf{r}) + {}^2\hat{\mathbf{b}}(t, \mathbf{r}), \quad (23)$$

$$\mathbf{v}(t, \mathbf{r}) = {}^1\mathbf{v}(t, \mathbf{r}) + {}^2\mathbf{v}(t, \mathbf{r}), \quad (24)$$

$$\mathbf{V}(t, \mathbf{r}) = {}^1\mathbf{V}(t, \mathbf{r}) + {}^2\mathbf{V}(t, \mathbf{r}) = [\mathbf{U}(t, \mathbf{r}) + {}^1\mathbf{v}(t, \mathbf{r})] + {}^2\mathbf{v}(t, \mathbf{r}). \quad (25)$$

Here, $\mathbf{V} = \mathbf{U} + \mathbf{v}$ is the total fluid velocity, and the Kolmogorov turbulent velocities \mathbf{U} are considered to be given and to be of the first order (Vainshtein 1970, Kulsrud & Anderson 1992).

Now, we substitute the above expansion formulas into equations (16)–(21). We find that the zero order equations are

$$\partial_t {}^0B_\alpha = 0, \quad \partial_t {}^0\hat{b}_\alpha = 0, \quad {}^0v_\alpha = 0, \quad {}^0V_\alpha = 0, \quad (26)$$

the first order equations are

$$\partial_t {}^1B_\alpha = {}^1V_{\alpha,\beta} {}^0B_\beta - {}^1V_\beta {}^0B_{\alpha,\beta}, \quad (27)$$

$$\partial_t {}^1\hat{b}_\alpha = {}^1V_{\alpha,\beta} {}^0\hat{b}_\beta - {}^1V_{\beta,\gamma} {}^0b_{\alpha\beta\gamma} - {}^1V_\beta {}^0\hat{b}_{\alpha,\beta}, \quad (28)$$

$$\partial_t {}^1v_\alpha = -{}^1P_{,\alpha} + 3\nu({}^0b_{\alpha\beta\mu\nu} {}^1v_{\mu,\nu})_{,\beta} + 3\nu({}^0b_{\alpha\beta\mu\nu} U_{\mu,\nu})_{,\beta} - (1/5)\nu\Delta U_\alpha, \quad (29)$$

$${}^1v_{\alpha,\alpha} = 0, \quad (30)$$

$${}^1V_\alpha = U_\alpha + {}^1v_\alpha, \quad (31)$$

and the second order equations are

$$\partial_t {}^2B_\alpha = {}^1V_{\alpha,\beta} {}^1B_\beta + {}^2V_{\alpha,\beta} {}^0B_\beta - {}^1V_\beta {}^1B_{\alpha,\beta} - {}^2V_\beta {}^0B_{\alpha,\beta}, \quad (32)$$

$$\partial_t {}^2\hat{b}_\alpha = {}^1V_{\alpha,\beta} {}^1\hat{b}_\beta + {}^2V_{\alpha,\beta} {}^0\hat{b}_\beta - {}^1V_{\beta,\gamma} {}^1b_{\alpha\beta\gamma} - {}^2V_{\beta,\gamma} {}^0b_{\alpha\beta\gamma} - {}^1V_\beta {}^1\hat{b}_{\alpha,\beta} - {}^2V_\beta {}^0\hat{b}_{\alpha,\beta}, \quad (33)$$

⁹A nonzero initial back-reaction velocity would lead to transients, which would be dissipated anyway.

¹⁰To be more formal, we need to introduce a dimensionless variable $\xi = t/\Delta t$, and to consider $U_{\alpha,\beta}\Delta t$ and $V_{\alpha,\beta}\Delta t$, which are dimensionless, as the expansion parameters.

$$\partial_t {}^2v_\alpha = -{}^2P_{,\alpha} + 3\nu[{}^0b_{\alpha\beta\mu\nu}{}^2v_{\mu,\nu} + {}^1b_{\alpha\beta\mu\nu}({}^1v_{\mu,\nu} + U_{\mu,\nu})]_{,\beta} - [{}^1v_\alpha U_\beta + U_\alpha {}^1v_\beta + {}^1v_\alpha {}^1v_\beta]_{,\beta}, \quad (34)$$

$${}^2v_{\alpha,\alpha} = 0, \quad (35)$$

$${}^2V_\alpha = {}^2v_\alpha. \quad (36)$$

Here, of course, the pressure is also expanded, $P = {}^1P + {}^2P$.

First, let us solve the first order equations (29) and (30). We Fourier transform these equations in space and in time, $\mathbf{r} \rightarrow \mathbf{k}$ and $t \rightarrow \omega$, by making use of the discrete and the continuous Fourier transformations respectively [see eq. (7)]. We have

$$(-i\omega + \Omega_{\text{rd}}){}^1\tilde{v}_{\mathbf{k}\alpha}(\omega) = -ik_\alpha {}^1\tilde{P}_{\mathbf{k}} + 3\nu ik_\beta \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} ik'_\nu [{}^1\tilde{v}_{\mathbf{k}'\mu} + \tilde{U}_{\mathbf{k}'\mu}] {}^0\tilde{b}_{\mathbf{k}''\alpha\beta\mu\nu} + (1/5)\nu k^2 \tilde{U}_{\mathbf{k}\alpha}, \quad (37)$$

$$k_\alpha {}^1\tilde{v}_{\mathbf{k}\alpha}(\omega) = 0, \quad (38)$$

where ${}^1\tilde{\mathbf{v}}_{\mathbf{k}}(\omega)$, ${}^1\tilde{P}_{\mathbf{k}}(\omega)$, $\tilde{\mathbf{U}}_{\mathbf{k}}(\omega)$ and ${}^0\tilde{b}_{\mathbf{k}\alpha\beta\mu\nu}$ are the Fourier coefficients. In the left-hand-side of equation (37) we add a damping term, $\Omega_{\text{rd}}{}^1\tilde{\mathbf{v}}_{\mathbf{k}}$, in order to account for the rotation of velocity vectors relative to the magnetic field unit vectors, see the discussion on page 15. We estimate *the effective rotational damping rate* Ω_{rd} on page 17, and show it is a constant. In general, it may be a function of $k = |\mathbf{k}|$ and of $(\hat{\mathbf{b}} \cdot \hat{\mathbf{k}})^2$.¹¹

Now, we multiply equation (37) on the left by tensor $\delta_{\gamma\alpha}^\perp = \delta_{\gamma\alpha} - \hat{k}_\gamma \hat{k}_\alpha$ to eliminate the pressure term by making use of the incompressibility condition (38). Interchanging indices, and using the symmetry of tensor ${}^0\tilde{b}_{\mathbf{k}''\alpha\beta\mu\nu}$ with respect to its spatial indices, we obtain

$${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta} {}^1\tilde{v}_{\mathbf{k}'\beta} = {}^1\mathcal{F}_{\mathbf{k}\alpha}, \quad (39)$$

$${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta} = (-i\omega + \Omega_{\text{rd}})\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha\beta} + 3\nu\delta_{\alpha\gamma}^\perp k_\mu k'_\nu {}^0\tilde{b}_{\mathbf{k}''\gamma\mu\nu\beta}, \quad \mathbf{k}'' = \mathbf{k} - \mathbf{k}', \quad (40)$$

$${}^1\mathcal{F}_{\mathbf{k}\alpha} = [{}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta} + (-i\omega + \Omega_{\text{rd}} + \nu k^2/5)\delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha\beta}] \tilde{U}_{\mathbf{k}'\beta}, \quad (41)$$

where we use convenient matrix notation, so that summation is implicitly assumed over repeated spatial indices and wave numbers. The matrix operator ${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta}(\omega)$ is of the zero order, while “the driving force” ${}^1\mathcal{F}_{\mathbf{k}\alpha}(\omega)$ is of the first order. If there exist inverse matrix ${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta}^{-1}$, then, using equations (31), (39) and (41), we obtain the Fourier coefficient of the first order total velocity ${}^1\mathbf{V}$,

$${}^1\tilde{V}_{\mathbf{k}\alpha}(\omega) = \tilde{U}_{\mathbf{k}\alpha} + {}^1\tilde{v}_{\mathbf{k}\alpha} = (-i\omega + \Omega_{\text{rd}} + \nu k^2/5) {}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta}^{-1} \tilde{U}_{\mathbf{k}'\beta}(\omega). \quad (42)$$

¹¹It depends on the square of $\hat{\mathbf{b}} \cdot \hat{\mathbf{k}}$ because the Braginskii viscosity is invariant under reflection $\hat{\mathbf{b}} \rightarrow -\hat{\mathbf{b}}$ (gyrating ions “do not care” about the exact direction of $\hat{\mathbf{b}}$).

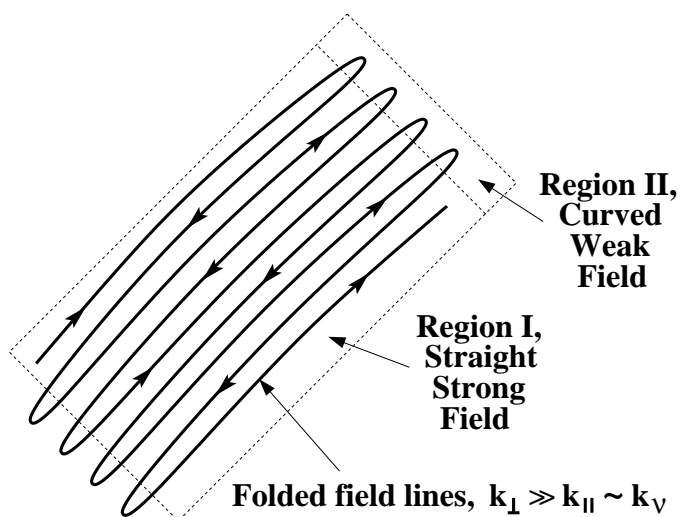


Fig. 2.— The folding structure of magnetic fields (for simplicity shown in two dimensions). The field is nearly straight and strong in Region I. The field is very curved but weak in Region II.

Now, let us consider an important case for which we can invert matrix ${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta}$, given by equation (40). Let $({}^0\mathbf{B}\cdot\nabla){}^0\mathbf{B} = 0$, i. e. let at zero time the magnetic field vary only in the direction perpendicular to itself. This is equivalent to the magnetic field lines being initially straight (no curvature), and ${}^0b_{\alpha\beta} = {}^0\hat{b}_\alpha {}^0\hat{b}_\beta \equiv \text{const.}$ This model of straight magnetic field lines is not as artificial as it seems at first glance, because of the following arguments. First, it is known that in the kinematic turbulent dynamo case the magnetic field lines have a folding pattern, shown in Figure 2, (Schekochihin *et al.* 2002; Maron & Cowley 2001). This folding pattern implies that in the bulk of the volume the field is strong and has small curvature, $k_\perp \gg k_\parallel \sim k_\nu$ (k_\parallel and k_\perp are the wave numbers parallel and perpendicular to the field lines), while in a small fraction of the volume the field is weak and curved, $k_\perp \sim k_\parallel \gg k_\nu$. The regions of weak and curved field, Region II in Figure 2, can be disregarded as long as we consider the volume averaged magnetic field energy and are not interested in the field curvature. As for the regions of strong magnetic field with small curvature, Region I in Figure 2, the field lines in these regions can be well approximated by the straight field lines on scales k which satisfy $k \gtrsim k_\parallel \sim k_\nu$. Now, we make assumption that even for the magnetized turbulent dynamo case the magnetic field has a folding structure similar to that for the kinematic turbulent dynamo case. This assumption will be our *first working hypothesis*. It is based on the simulations of Maron and Cowley (2001), who found good indications for the magnetic field folding pattern in their numerical simulations of MHD turbulence with the Braginskii viscosity. Unfortunately, our calculational methods are not adequate to theoretically justify our first working hypothesis because of complications of the field curvature calculations. However, there exist a simple *reductio ad absurdum* theoretical argument, supporting the hypothesis, which is as follows. If the magnetic field were not folded, i. e. if $k_\perp \sim k_\parallel \gtrsim k_\nu$ everywhere in space, then the field would be isotropically tangled on viscous and subviscous scales. In this case the Braginskii viscosity could be averaged over the field direction, and it should quickly reduce to the isotropic effective viscosity $\nu_{\text{eff}} = \nu/5$ [see eq. (6)]. As a result, in this case the magnetized dynamo should possess the properties of the kinematic dynamo with this effective viscosity, and would develop the folding structure of the magnetic field lines.

Note that at a given fixed spatial point of the Eulerian system of coordinates the field curvature changes in time. Thus, even if the field is initially straight at this point, it may not remain straight in the future. In other words, as time goes on, sometimes the spatial point belongs to a region I and sometimes to a region II (this reflects the time intermittency and spatial convection of the curvature). However, we are not interested in the magnetic energy evolution at each point of space, but instead, we are interested in the evolution of the total (spatially averaged) magnetic energy. Therefore, at any arbitrary moment of time we consider only those points of space that belong to all regions I at this time moment. These points give the dominant contribution to the total magnetic energy, and at these points the field lines can be considered to be straight. We apply our mathematical model of straight field lines to these points, and we find the change (the time derivative) of the total magnetic energy at the time moment we consider (see Section 4). Because the moment of time is arbitrary, our final results give the correct evolution of the total magnetic energy in time.

Thus, from our first working hypothesis ${}^0b_{\alpha\beta} \equiv \text{const}$, we have ${}^0\tilde{b}_{\mathbf{k}'',\gamma\mu\nu\beta} = \delta_{\mathbf{k}'',0} {}^0\hat{b}_\gamma {}^0\hat{b}_\mu {}^0\hat{b}_\nu {}^0\hat{b}_\beta$, and can easily invert matrix (40),

$${}^0\mathcal{M}_{\mathbf{k}\alpha,\mathbf{k}'\beta}^{-1} = \frac{\delta_{\mathbf{k},\mathbf{k}'}}{-i\omega + \Omega_{\text{rd}}} \left[\delta_{\alpha\beta} - \frac{3\nu k^2 \mu^2 (1 - \mu^2)}{-i\omega + \Omega_{\text{rd}} + 3\nu k^2 \mu^2 (1 - \mu^2)} \frac{\delta_{\alpha\gamma}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\beta}{1 - \mu^2} \right]. \quad (43)$$

$$\mu \stackrel{\text{def}}{=} {}^0\hat{\mathbf{b}} \cdot \hat{\mathbf{k}}. \quad (44)$$

We substitute this formula into equation (42) and obtain

$${}^1\tilde{V}_{\mathbf{k}\alpha}(\omega) = {}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) + {}^1\tilde{V}''_{\mathbf{k}\alpha}(\omega), \quad (45)$$

$${}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) = \frac{-i\omega + \Omega_{\text{rd}} + \bar{\Omega}}{-i\omega + \Omega_{\text{rd}}} \left[\delta_{\alpha\beta} - \frac{\delta_{\alpha\gamma}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\beta}{1 - \mu^2} \right] \tilde{U}_{\mathbf{k}\beta}(\omega), \quad (46)$$

$${}^1\tilde{V}''_{\mathbf{k}\alpha}(\omega) = \frac{-i\omega + \Omega_{\text{rd}} + \bar{\Omega}}{-i\omega + \Omega_{\text{rd}} + 2\bar{\Omega}} \frac{\delta_{\alpha\gamma}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\beta}{1 - \mu^2} \tilde{U}_{\mathbf{k}\beta}(\omega), \quad (47)$$

where, we introduce the following notations for the viscous damping frequencies

$$\bar{\Omega} \stackrel{\text{def}}{=} (1/5)\nu k^2 = \nu_{\text{eff}} k^2, \quad \Omega \stackrel{\text{def}}{=} (3/2)\nu k^2 \mu^2 (1 - \mu^2) = (15/2)\bar{\Omega} \mu^2 (1 - \mu^2). \quad (48)$$

The frequency Ω depends on μ^2 because of the anisotropy of the Braginskii viscous stress tensor. The frequency $\bar{\Omega}$ is equal to Ω averaged over μ , and represents the averaged rate of the Braginskii viscous dissipation [see eq. (6)].

Next, we calculate $\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) \rangle$ and $\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{\mathbf{k}'\beta}(t') \rangle$, which are the ensemble averages of the ${}^1\mathbf{V}$'s over all possible realizations of the turbulent motions. First, using formulas (8), (9) and (45)–(47), we find

$$\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(\omega) \rangle = 0, \quad (49)$$

$$\begin{aligned} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}_{\mathbf{k}'\beta}(\omega') \rangle &= \langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}'_{\mathbf{k}'\beta}(\omega') \rangle + \langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}''_{\mathbf{k}'\beta}(\omega') \rangle \\ &= J_{\omega k} \delta_{\mathbf{k}', -\mathbf{k}} \delta(\omega' + \omega) \left[\tilde{H}_F(\omega; \bar{\Omega} + \Omega_{\text{rd}}, \Omega_{\text{rd}}) \left(\delta_{\alpha\beta}^\perp - \frac{\delta_{\alpha\gamma}^\perp \delta_{\beta\tau}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\tau}{1 - \mu^2} \right) \right. \\ &\quad \left. + \tilde{H}_F(\omega; \bar{\Omega} + \Omega_{\text{rd}}, 2\bar{\Omega} + \Omega_{\text{rd}}) \frac{\delta_{\alpha\gamma}^\perp \delta_{\beta\tau}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\tau}{1 - \mu^2} \right], \end{aligned} \quad (50)$$

$$\tilde{H}_F(\omega; q_1, q_2) \stackrel{\text{def}}{=} \frac{\omega^2 + q_1^2}{\omega^2 + q_2^2}. \quad (51)$$

Note that $\langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(\omega) {}^1\tilde{V}''_{\mathbf{k}'\beta}(\omega') \rangle = 0$, this implies that ${}^1\tilde{\mathbf{V}}_{\mathbf{k}}'$ and ${}^1\tilde{\mathbf{V}}_{\mathbf{k}}''$ are perpendicular on average. Second, we apply the inverse Fourier transformations in time, $\omega, \omega' \rightarrow t, t'$, to equations (49), (50). We have for the required correlation functions of the total turbulent velocities

$$\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) \rangle = 0, \quad (52)$$

$$\begin{aligned}
\langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t){}^1\tilde{V}_{\mathbf{k}'\beta}(t') \rangle &= \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t){}^1\tilde{V}_{-\mathbf{k}'\beta}^*(t') \rangle = \langle {}^1\tilde{V}'_{\mathbf{k}\alpha}(t){}^1\tilde{V}'_{\mathbf{k}'\beta}(t') \rangle + \langle {}^1\tilde{V}''_{\mathbf{k}\alpha}(t){}^1\tilde{V}''_{\mathbf{k}'\beta}(t') \rangle \\
&= \delta_{\mathbf{k}',-\mathbf{k}} \left[H_F(t-t'; \bar{\Omega} + \Omega_{\text{rd}}, \Omega_{\text{rd}}) \left(\delta_{\alpha\beta}^\perp - \frac{\delta_{\alpha\gamma}^\perp \delta_{\beta\tau}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\tau}{1-\mu^2} \right) \right. \\
&\quad \left. + H_F(t-t'; \bar{\Omega} + \Omega_{\text{rd}}, 2\Omega + \Omega_{\text{rd}}) \frac{\delta_{\alpha\gamma}^\perp \delta_{\beta\tau}^\perp {}^0\hat{b}_\gamma {}^0\hat{b}_\tau}{1-\mu^2} \right], \tag{53}
\end{aligned}$$

where the function

$$H_F(t-t'; q_1, q_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} J_{\omega k} \tilde{H}_F(\omega; q_1, q_2) e^{-i\omega(t-t')} d\omega = \frac{J_{0k}}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2 + q_1^2}{\omega^2 + q_2^2} \frac{\cos[\omega(t-t')]}{1 + \tau^2 \omega^2} d\omega \tag{54}$$

is the inverse Fourier transform of function $J_{\omega k} \tilde{H}_F(\omega; q_1, q_2)$, and depends only on the absolute value of the time difference $t - t'$. In equation (54) we use formulas (11) and (51).

The main unknown quantity is the effective rotational damping coefficient Ω_{rd} , which we introduced in equation (37) as the $\Omega_{\text{rd}} {}^1\tilde{\mathbf{v}}_{\mathbf{k}}$ term. Let us consider it in more detail. We start with the calculation of the ensemble average of ${}^1\mathbf{V}$ squared¹². We have

$$\langle {}^1\mathbf{V}^2 \rangle = \langle {}^1V_\alpha(t, \mathbf{r}) {}^1V_\alpha(t, \mathbf{r}) \rangle = \sum_{\mathbf{k}, \mathbf{k}'} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t) {}^1\tilde{V}_{\mathbf{k}'\alpha}(t) \rangle e^{i(\mathbf{k}+\mathbf{k}')\mathbf{r}} = \sum_{\mathbf{k}} \langle |{}^1\tilde{\mathbf{V}}_{\mathbf{k}}|^2 \rangle \tag{55}$$

[see eq. (53)]. Using equations (53) and (54), we obtain

$$\begin{aligned}
\langle |{}^1\tilde{\mathbf{V}}_{\mathbf{k}}|^2 \rangle &= H_F(0; \bar{\Omega} + \Omega_{\text{rd}}, \Omega_{\text{rd}}) + H_F(0; \bar{\Omega} + \Omega_{\text{rd}}, 2\Omega + \Omega_{\text{rd}}) \\
&= \frac{J_{0k}}{\tau} + \frac{J_{0k}}{2} \frac{(\bar{\Omega} + \Omega_{\text{rd}})^2 - \Omega_{\text{rd}}^2}{\Omega_{\text{rd}}(1 + \tau\Omega_{\text{rd}})} + \frac{J_{0k}}{2} \frac{(\bar{\Omega} + \Omega_{\text{rd}})^2 - (2\Omega + \Omega_{\text{rd}})^2}{(2\Omega + \Omega_{\text{rd}})[1 + \tau(2\Omega + \Omega_{\text{rd}})]}. \tag{56}
\end{aligned}$$

According to equations (55) and (56), we see that if we set the rotational damping rate Ω_{rd} to zero, then the ensemble average of the first order velocity squared would become infinite, $\langle {}^1\mathbf{V}^2 \rangle \rightarrow \infty$ as $\Omega_{\text{rd}} \rightarrow 0$! Let us try to understand this divergence problem, and see how we can avoid it.

First, following the derivation of formula (56) from equation (53), it is easy to see that the divergence of $\langle {}^1\mathbf{V}^2 \rangle$ occurs only due to the divergence of the ${}^1\tilde{V}'_{\mathbf{k}\alpha}$ modes. In other words, $\langle |{}^1\tilde{\mathbf{V}}'_{\mathbf{k}}|^2 \rangle \rightarrow \infty$ as $\Omega_{\text{rd}} \rightarrow 0$, while $\langle |{}^1\tilde{\mathbf{V}}''_{\mathbf{k}}|^2 \rangle$ stays finite¹³. Second, let us refer to equations (46) and (47). On one hand, we have $\hat{k}_\alpha {}^1\tilde{V}'_{\mathbf{k}\alpha} = 0$ and $\hat{k}_\alpha {}^1\tilde{V}''_{\mathbf{k}\alpha} = 0$, as it should be because plasma velocities are incompressible. On the other hand, we have ${}^0\hat{b}_\alpha {}^1\tilde{V}'_{\mathbf{k}\alpha} = 0$ and ${}^0\hat{b}_\alpha {}^1\tilde{V}''_{\mathbf{k}\alpha} \neq 0$. Thus, the divergent velocity modes, ${}^1\tilde{\mathbf{V}}'_{\mathbf{k}}$, are perpendicular to both vectors $\hat{\mathbf{k}}$ and ${}^0\hat{\mathbf{b}}$. At the same time, the other, non-divergent modes, ${}^1\tilde{\mathbf{V}}''_{\mathbf{k}}$, have nonzero components along ${}^0\hat{\mathbf{b}}$. Third, let us calculate the ensemble

¹²Note that $\langle {}^1\mathbf{V} \rangle = 0$ because there is no preferred direction. Of course, there is a preferred axis in space, which is along the magnetic field unit vector, see also the footnote 11 on page 11.

¹³Of course, in the degenerate cases, when $\hat{\mathbf{k}} \perp {}^0\hat{\mathbf{b}}$ (i. e. $\mu^2 = 0$) or $\hat{\mathbf{k}} \parallel {}^0\hat{\mathbf{b}}$ (i. e. $1 - \mu^2 = 0$), both ${}^1\tilde{\mathbf{V}}'_{\mathbf{k}}$ and ${}^1\tilde{\mathbf{V}}''_{\mathbf{k}}$ modes become infinite as $\Omega_{\text{rd}} \rightarrow 0$.

averaged Braginskii viscous dissipation into heat (Braginskii 1965). We have

$$\begin{aligned}\langle Q_{\text{vis}} \rangle &= \langle \pi_{\alpha\beta} V_{\alpha,\beta} \rangle = -3\nu \langle {}^1V_{\alpha,\beta} {}^1V_{\gamma,\tau} \rangle {}^0b_{\alpha\beta\gamma\tau} = 3\nu \sum_{\mathbf{k}, \mathbf{k}'} k_\beta k'_\tau \langle {}^1\tilde{V}_{\mathbf{k}\alpha} {}^1\tilde{V}_{\mathbf{k}'\gamma} \rangle {}^0b_{\alpha\beta\gamma\tau} e^{i(\mathbf{k}+\mathbf{k}')\mathbf{r}} \\ &= -3\nu \sum_{\mathbf{k}} k^2 \mu^2 {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}\alpha}'' {}^1\tilde{V}_{-\mathbf{k}\gamma}'' \rangle {}^0\hat{b}_\gamma = -2 \sum_{\mathbf{k}} \Omega H_F(0; \bar{\Omega} + \Omega_{\text{rd}}, 2\Omega + \Omega_{\text{rd}}),\end{aligned}\quad (57)$$

where we keep only the first order velocities, and make use of formulas ${}^0b_{\alpha\beta\gamma\tau} = \text{const}$, ${}^0\hat{b}_\alpha {}^1\tilde{V}_{\mathbf{k}\alpha}' = 0$, and of equations (1), (44), (48), (53). We see that only the ${}^1\tilde{\mathbf{V}}_{\mathbf{k}}''$ velocity modes are dissipated by the Braginskii viscous forces, and the dissipation is proportional to $2\Omega = 3\nu k^2 \mu^2 (1 - \mu^2)$.

As a result, we conclude that the divergence of the first order velocities happens because the ${}^1\tilde{\mathbf{V}}_{\mathbf{k}}'$ velocity modes, which are perpendicular to ${}^0\hat{\mathbf{b}}$, are not damped by the Braginskii viscous dissipation. On the other hand, it is clear that the ${}^1\tilde{\mathbf{V}}_{\mathbf{k}}'$ modes can not be infinite. What are the physical mechanisms which limit them? To answer this question, let us note that velocity modes are non-linearly coupled with each other via the inertial term, $(\mathbf{V} \cdot \nabla)\mathbf{V}$, of the MHD equation (2). This non-linear coupling transforms the divergent ${}^1\tilde{\mathbf{V}}_{\mathbf{k}}'$ velocity modes into other modes, which are then viscously dissipated. This transformation can be viewed as a continuous rotation of velocity vectors relative to the magnetic field unit vector $\hat{\mathbf{b}}$. Indeed, at a given point of space we can go to a reference frame that rotates together with $\hat{\mathbf{b}}$. In this rotating frame there exist Coriolis forces, which act on velocity vectors and force them to rotate relative to the non-rotating field vector. These Coriolis forces are caused by the non-linear coupling of velocity modes via the inertial term. As a result, a divergent velocity mode, perpendicular to $\hat{\mathbf{b}}$ and not viscously damped, eventually rotates out of its initial direction and is transformed into a damped mode. We call this process the “effective rotational damping”. Of course, it operates only on the viscous scales, on which the Braginskii viscous dissipation is significant. On larger scales the viscous dissipation is small and the rotation of velocities does not make any difference.

We could regard Ω_{rd} as a parameter to be determined by numerical simulations. Nevertheless, it is of interest to attempt to estimate it from a physical argument. First, the square of the angular velocity of the rotation can be estimated as $\omega_{\text{rot}}^2 \sim (1/3)\tau_\nu^{-2}$, where τ_ν is the velocity decorrelation time on the viscous scale. The factor 1/3 in this equation comes from the fact that only one of the three angular velocity components contributes to the deviation of a divergent velocity mode from its original direction perpendicular to the field unit vector $\hat{\mathbf{b}}$ ¹⁴. Second, the typical Braginskii viscous damping rate is clearly about $1/\tau_\nu \sim \nu_{\text{eff}} k_\nu^2$, and is larger than the angular velocity of the rotation. Let assume it is much larger, $\omega_{\text{rot}}^2 \tau_\nu^2 \sim 1/3 \ll 1$. Then we can suppose that after the divergent velocity mode, in a time interval Δt , rotates by an angle $\Delta\phi$ relative to its original direction perpendicular to $\hat{\mathbf{b}}$, only the projection of the velocity mode on the plane perpendicular to $\hat{\mathbf{b}}$ survives, and all other velocity components are immediately viscously dissipated. As a result,

¹⁴This is the component along the vector product $\hat{\mathbf{b}} \times \mathbf{V}$.

the effective rotational damping rate, Ω_{rd} , can be estimated as follows.

$$\begin{aligned} \frac{dV}{dt} &\sim \frac{\Delta V}{\Delta t} \sim \frac{V(\cos \Delta\phi - 1)}{\Delta t} \sim -\frac{V}{2\Delta t} \Delta\phi^2 \sim -\frac{V}{2\Delta t} (\omega_{\text{rot}}\tau_\nu)^2 \frac{\Delta t}{\tau_\nu} \\ &\sim -(1/6)\tau_\nu^{-1}V \sim (1/6)\nu_{\text{eff}}k_\nu^2 V \sim (1/30)\nu k_\nu^2 V = -\Omega_{\text{rd}}V, \\ \Omega_{\text{rd}} &= (1/6)\nu_{\text{eff}}k_\nu^2 = (1/30)\nu k_\nu^2. \end{aligned} \quad (58)$$

To obtain the last result in the first line of equation (58), we use the random-walk approximation for the estimate of $\Delta\phi^2$.¹⁵

Let us summarize our discussion of the effective rotational damping. As we said, this physical damping is associated with the non-linear coupling of velocity modes, which leads to the rotation of velocities relative to the magnetic field vectors. The effective rotational damping is very important because it limits the velocity modes which are perpendicular to the magnetic field vectors, and therefore, are undamped by the Braginskii viscous forces. The non-linear coupling of the velocity modes is hard to deal with directly. In particular, the MHD non-linear inertial terms do not appear in our first-order equation (37). As a result, to avoid the divergence problem for the undamped velocity modes, we incorporate the non-linear mode coupling and the associated rotational damping into our equations in a simple way, as the $\Omega_{\text{rd}}\mathbf{\tilde{v}}_{\mathbf{k}}$ damping term in the left-hand-side of equation (37). This is our *second working hypothesis*. Note, that the $\Omega_{\text{rd}}\mathbf{\tilde{v}}_{\mathbf{k}}$ term is isotropic, and therefore, it damps not only the divergent velocity modes, perpendicular to the field vector, but all velocity modes. This is not a serious problem though, because the rotational damping is smaller than the Braginskii damping by a factor $\sim 1/6$ [see eq. (59)], and our results should be valid within a factor of order unity. Also note that on scales larger than the viscous scale the rotational damping does not operate. However, on those large scales the turbulence is Kolmogorov, $\mathbf{V} = \mathbf{U}$, and the back reaction velocities \mathbf{v} are zero anyway¹⁶.

So far we have considered only the first order velocity, $\mathbf{\tilde{v}}_1$, and have found its statistics, given by equations (52)–(54) and (59). In order to find the second order velocity, $\mathbf{\tilde{v}}_2$, we need to solve the complicated second order equations (34)–(36). Fortunately, we will need only the ensemble average of the second order velocity, $\langle \mathbf{\tilde{v}}_2 \rangle$. It turns out that in our case of a straight initial field, ${}^0b_{\alpha\beta} = \text{const}$, which we consider here, this average is zero,

$$\langle \mathbf{\tilde{v}}_2(t) \rangle = 0, \quad (60)$$

(Malyshkin 2001). The reason for this simple result is that ${}^0b_{\alpha\beta}$ is constant in space, and the Kolmogorov turbulence, \mathbf{U} , is statistically homogeneous. Therefore, the ensemble averages of the terms in the brackets [...] in equation (34) are constant in space, and their spatial derivatives are zero. As a result, the ensemble averaged velocity $\langle \mathbf{\tilde{v}}_2 \rangle = \langle \mathbf{\tilde{v}}_2 \rangle$ is also zero.

¹⁵We again assume that $\omega_{\text{rot}}^2\tau_\nu^2 \sim 1/3 \ll 1$. Next we choose such time interval $\Delta t \gg \tau_\nu$, that $\Delta\phi \sim \omega_{\text{rot}}\Delta t \ll 1$. Then we can simultaneously expand the $\cos \Delta\phi$ in eq. (58) and use the random-walk approximation to estimate $\Delta\phi^2$.

¹⁶In particular, note that the driving force (41), which is $\propto k^2$, becomes small on large scales.

To conclude this section, let us integrate equation (54) in time, and obtain the formulas

$$\begin{aligned} \int_0^t dt' \int_0^{t'} H_F(t' - t'') dt'' &= \frac{J_{0k}}{\pi} \int_{-\infty}^{\infty} \frac{\omega^2 + q_1^2}{\omega^2 + q_2^2} \frac{\sin^2(\omega t/2)}{1 + \tau^2 \omega^2} \frac{d\omega}{\omega^2} = \frac{J_{0k}}{2} \left[t - \tau(1 - e^{-t/\tau}) \right] \\ &+ \frac{J_{0k}}{2} \frac{q_1^2 - q_2^2}{1 - \tau^2 q_2^2} \left[\frac{q_2 t - 1 + e^{-q_2 t}}{q_2^3} - \tau^2 t + \tau^3(1 - e^{-t/\tau}) \right] \rightarrow \frac{J_{0k}}{2} \frac{q_1^2}{q_2^2} t, \end{aligned} \quad (61)$$

$$\int_0^t dt' \int_0^{t'} H_F(t' - t'') dt'' = 2 \int_0^t dt' \int_0^{t'} H_F(t' - t'') dt'' \rightarrow J_{0k} \frac{q_1^2}{q_2^2} t, \quad (62)$$

which we will use below. Here, the integrals over ω can be done by closing the integration contours in the complex plane and by evaluating the residues. The final answers in these formulas, written after the right arrows “ \rightarrow ”, give the results in the limit $t \gg \tau, q_2^{-1}$.

4. Energy Spectrum of Random Magnetic Fields

In this section we use equations (52), (53) and (60), which give the statistics of the turbulent velocities, to derive the evolution of the magnetic energy in the magnetized turbulent dynamo theory.

4.1. The Growth of the Total Magnetic Energy

The volume averaged and ensemble averaged magnetic energy per unit mass is

$$\mathcal{E} \stackrel{\text{def}}{=} \frac{1}{L^3} \int \frac{\langle B^2 \rangle}{8\pi\rho} d^3\mathbf{r} = \frac{1}{8\pi\rho} \langle \widetilde{B}_{\mathbf{k}=0}^2 \rangle, \quad (63)$$

where ρ is the plasma density, and $\widetilde{B}_{\mathbf{k}=0}$ is the $\mathbf{k} = 0$ Fourier coefficient of the magnetic field strength squared, B^2 . To find $\mathcal{E}(t)$, it is convenient to introduce a symmetric tensor $B_{\alpha\beta} \stackrel{\text{def}}{=} B_\alpha B_\beta = B^2 b_{\alpha\beta}$. The differential equation for $B_{\alpha\beta}$ follows from equation (19),

$$\partial_t B_{\alpha\beta} = B_\alpha \partial_t B_\beta + B_\beta \partial_t B_\alpha = V_{\alpha,\gamma} B_{\beta\gamma} + V_{\beta,\gamma} B_{\alpha\gamma} - V_\gamma B_{\alpha\beta,\gamma}. \quad (64)$$

Now, we simultaneously solve this equation and equation (20) by making use of the quasi-linear expansion procedure, described in Section 3. First, we write

$$B^2(t) = {}^0B^2 + {}^1B^2(t) + {}^2B^2(t), \quad B_{\alpha\beta}(t) = {}^0B_{\alpha\beta} + {}^1B_{\alpha\beta}(t). \quad (65)$$

Second, we substitute these expansion formulas into equations (20) and (64). We find that the first order equations are

$$\partial_t {}^1B^2 = 2 {}^1V_{\alpha,\beta} {}^0B_{\alpha\beta} - {}^1V_\beta {}^0B^2_{,\beta}, \quad (66)$$

$$\partial_t {}^1B_{\alpha\beta} = {}^1V_{\alpha,\gamma} {}^0B_{\beta\gamma} + {}^1V_{\beta,\gamma} {}^0B_{\alpha\gamma} - ({}^1V_\gamma {}^0B_{\alpha\beta})_{,\gamma}, \quad (67)$$

and the second order equation for ${}^2B^2(t)$ is

$$\partial_t {}^2B^2 = {}^2V_{\alpha,\beta} {}^1B_{\alpha\beta} + {}^2V_{\alpha,\beta} {}^0B_{\alpha\beta} - ({}^1V_{\beta} {}^1B^2)_{,\beta} - ({}^2V_{\beta} {}^0B^2)_{,\beta}. \quad (68)$$

Third, we integrate equation (66) in time (with the zero initial conditions) and ensemble average the result. Using equation (52), we obviously obtain $\langle {}^1B^2(t) \rangle = 0$. Fourth, we integrate equation (67) in time, and then Fourier transform the result in space, $\mathbf{r} \rightarrow \mathbf{k}$. We have

$${}^1\tilde{B}_{\mathbf{k}\alpha\beta}(t) = i \left[k'_{\gamma} (\delta_{\alpha\tau} {}^0b_{\beta\gamma} + \delta_{\beta\tau} {}^0b_{\alpha\gamma}) - k_{\tau} {}^0b_{\alpha\beta} \right] \int_0^t \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} {}^1\tilde{V}_{\mathbf{k}'\tau}(t') {}^0\tilde{B}^2_{\mathbf{k}''} dt'. \quad (69)$$

Here, we use $B_{\alpha\beta} = B^2 b_{\alpha\beta}$ and ${}^0b_{\alpha\beta} = \text{const.}$ (This last formula is our first working hypothesis). Fifth, we integrate the second order equation (68) in time. Then we ensemble average the result and Fourier transform it in space, setting \mathbf{k} to zero. Using equations (53), (60) and (69), we obtain

$$\begin{aligned} \langle {}^2\tilde{B}^2_{\mathbf{k}=0}(t) \rangle &= 2i \int_0^t \sum_{\mathbf{k}} k_{\beta} \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t') {}^1\tilde{B}_{-\mathbf{k}\alpha\beta}(t') \rangle dt' \\ &= 2 {}^0\tilde{B}^2_{\mathbf{k}=0} \int_0^t dt' \int_0^{t'} \sum_{\mathbf{k}} \mu^2 k^2 \langle {}^1\tilde{V}_{\mathbf{k}\alpha}(t') {}^1\tilde{V}_{-\mathbf{k}\alpha}(t'') \rangle dt'' = 2 {}^0\tilde{B}^2_{\mathbf{k}=0} \sum_{\mathbf{k}} \mu^2 k^2 \\ &\quad \times \int_0^t dt' \int_0^{t'} \left[H_F(t' - t''; \bar{\Omega} + \Omega_{\text{rd}}, \Omega_{\text{rd}}) + H_F(t' - t''; \bar{\Omega} + \Omega_{\text{rd}}, 2\Omega + \Omega_{\text{rd}}) \right] dt'' \\ &= 2\gamma t {}^0\tilde{B}^2_{\mathbf{k}=0}, \end{aligned} \quad (70)$$

where

$$\gamma = \pi \left(\frac{L}{2\pi} \right)^3 \int_0^{\infty} k^4 J_{0k} dk \int_{-1}^1 \mu^2 (1 + \bar{\Omega}/\Omega_{\text{rd}})^2 \left[1 + (1 + 2\Omega/\Omega_{\text{rd}})^{-2} \right] d\mu, \quad (71)$$

$$\bar{\Omega}/\Omega_{\text{rd}} = 6k^2/k_{\nu}^2, \quad 2\Omega/\Omega_{\text{rd}} = 90(k^2/k_{\nu}^2)\mu^2(1 - \mu^2). \quad (72)$$

Here, we also use equation (61) in the limit $t \gg \tau$, replace the summation over \mathbf{k} by integration, making use of $d^3\mathbf{k} = 2\pi k^2 dk d\mu$, and use equations (48), (59).

Now, we derive the differential equation for the averaged magnetic energy \mathcal{E} . Following Kulsrud and Anderson (1992), we choose t small enough for the quasi-linear expansion to be valid, but large enough for the limit $t \gg \tau$ to be satisfied. This is very similar to an assumption that the turbulent velocities are δ -correlated in time (Kazantsev 1968), the assumption generally used in the dynamo theories (see the discussion in the end of this section). As a result, using equations (65) and (70), we obtain

$$\partial_t \langle \tilde{B}^2_{\mathbf{k}=0} \rangle = \frac{1}{t} \langle \tilde{B}^2_{\mathbf{k}=0}(t) - \tilde{B}^2_{\mathbf{k}=0}(0) \rangle = \frac{1}{t} \langle {}^1\tilde{B}^2_{\mathbf{k}=0} + {}^2\tilde{B}^2_{\mathbf{k}=0} \rangle = \frac{1}{t} \langle {}^2\tilde{B}^2_{\mathbf{k}=0} \rangle = 2\gamma {}^0\tilde{B}^2_{\mathbf{k}=0}, \quad (73)$$

and using equation (63), we finally obtain

$$\partial \mathcal{E} / \partial t = 2\gamma \mathcal{E}. \quad (74)$$

According to this last equation, the magnetic energy grows exponentially in time, the same way as it does in the kinematic dynamo theory (Kulsrud & Anderson 1992). However, the growth rate γ , given by equation (71) in the magnetized turbulent dynamo case, is different from the growth rate γ_o in the kinematic dynamo case. The latter can easily be obtained by taking the limit $\Omega_{\text{rd}} \rightarrow \infty$ in equation (71) and by changing the viscous cutoff wave number in equation (13) to the standard one, $k_\nu = (U_0/k_0\nu)^{3/4}k_0$. In this limit, the back-reaction velocities are zero (because they are totally damped), the turbulence is Kolmogorov, and equation (71) reduces to the corresponding formula of Kulsrud and Anderson (1992), as one might expect.

The integrals in equation (71) can be carried out numerically (Malyshkin 2001). The result is

$$\gamma \approx 8.5 (U_0 L/\nu)^{1/2} (U_0/L), \quad \gamma^{-1} \approx 10^6 \text{ yrs } (\xi/10)^{1/2} (L/0.2 \text{ Mpc})^{3/2}, \quad (75)$$

$$\gamma t_{\text{collapse}} \approx \gamma (L/U_0) \approx 10^3 (\xi/10)^{-1/2} (M/10^{12} \text{ M}_\odot)^{-1/2}, \quad (76)$$

$$\gamma/\gamma_o \approx 10. \quad (77)$$

Here, t_{collapse} is the protogalaxy collapse time, ξ is the ratio of the total mass M to the baryon mass, L is the protogalaxy size, and the temperature is assumed to be virial. Equation (77) predicts that the magnetic energy growth rate in the magnetized dynamo theory is up to ten times larger than that in the kinematic dynamo theory. Two different effects contribute to this difference. First, the effective viscosity in the magnetized dynamo case is smaller than the molecular viscosity, $\nu_{\text{eff}} = \nu/5$. This effect makes the growth rate larger by a factor of $\sqrt{5}$ (this factor was included in Kulsrud *et al.* 1997). The rest of the contribution comes from the local anisotropy of the turbulent velocities in a strongly magnetized plasma.

Note that according to equations (12) and (75), the magnetic field growth time, $\gamma^{-1} = 0.12(U_0 L/\nu)^{-1/2}(L/U_0)$, is approximately 40% smaller than the eddy turnover time on the viscous scale, $\tau(k_\nu) \sim 0.18(U_0 L/\nu)^{-1/2}(L/U_0)$.¹⁷ Thus, the quasi-linear expansion of the MHD equations in time may not be fully compatible with the assumption of the δ -time correlation of the turbulent velocities. On the other hand, the effects of the finite velocity correlation time should decrease the magnetic energy growth rate by a factor of order two (this reduction was found in the kinematic turbulent dynamo theory by Schekochihin & Kulsrud, 2001). As a result of this reduction, the expansion should become better justified. Thus, including the finite time correlation effects into our theory would be important but not vital, since our calculation predicts a very large energy growth rate anyway [see eqs. (75)–(77)].

¹⁷In the kinematic dynamo theory $\tau(k_\nu) \sim (1/3)\gamma^{-1}$ (Kulsrud & Anderson 1992).

4.2. The Mode Coupling Equation for the Magnetic Energy Spectrum

The ensemble averaged magnetic energy spectrum is

$$M(t, k) = \frac{1}{4\pi\rho} \left(\frac{L}{2\pi} \right)^3 \int k^2 \langle |\tilde{\mathbf{B}}(t, \mathbf{k})|^2 \rangle d^2\hat{\mathbf{k}}, \quad (78)$$

where the integration is carried out over all directions of $\hat{\mathbf{k}} = \mathbf{k}/k$, and $\tilde{\mathbf{B}}_{\mathbf{k}}$ is the Fourier coefficient of the magnetic field, \mathbf{B} . The total magnetic energy, given by equation (63), is obviously

$$\mathcal{E} = \frac{1}{2} \int_0^\infty M(t, k) dk. \quad (79)$$

To find the evolution of $M(t, k)$, we use the quasi-linear expansion formula (22) for \mathbf{B} . We write the ensemble averaged square of the magnetic field Fourier coefficient, up to the second order,

$$\begin{aligned} \langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle &= |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 + [\langle \tilde{B}_{\mathbf{k}\alpha}(t) \rangle \tilde{B}_{\mathbf{k}\alpha}^*(t) + \text{c.c.}] + \langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle + [\langle \tilde{B}_{\mathbf{k}\alpha}(t) \rangle \tilde{B}_{\mathbf{k}\alpha}^*(t) + \text{c.c.}] \\ &= |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 + \langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle + [\langle \tilde{B}_{\mathbf{k}\alpha}(t) \rangle \tilde{B}_{\mathbf{k}\alpha}^*(t) + \text{c.c.}]. \end{aligned} \quad (80)$$

Here, c.c. is the complex conjugate, and we use the fact that the ensemble averaged first order magnetic field is zero, $\langle \tilde{\mathbf{B}}_{\mathbf{k}} \rangle = 0$, [this follows from eqs. (27) and (52)]. We calculate the two last terms in the right-hand-side of equation (80) separately.

We start with calculation of the $\langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle$ term. First, we integrate equation (27) in time, and then Fourier transform the result in space, $\mathbf{r} \rightarrow \mathbf{k}$. We have

$$\tilde{B}_{\mathbf{k}\chi}(t) = ik_\gamma (\delta_{\chi\alpha} \delta_{\gamma\delta} - \delta_{\gamma\alpha} \delta_{\chi\delta}) \int_0^t \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} \tilde{V}_{\mathbf{k}''\alpha}(t') \tilde{B}_{\mathbf{k}'\delta} dt', \quad (81)$$

where we use the divergence free conditions $k'_\alpha \tilde{B}_{\mathbf{k}'\alpha} = 0$ and $k''_\alpha \tilde{V}_{\mathbf{k}''\alpha} = 0$. Using this equation and its complex conjugate, we obtain

$$\begin{aligned} \langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle &= k_\gamma k_\tau (\delta_{\chi\alpha} \delta_{\gamma\delta} - \delta_{\gamma\alpha} \delta_{\chi\delta}) (\delta_{\chi\beta} \delta_{\tau\eta} - \delta_{\tau\beta} \delta_{\chi\eta}) \\ &\quad \times \int_0^t \int_0^t \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} \sum_{\substack{\mathbf{k}''' \\ \mathbf{k}^{\text{iv}} = \mathbf{k} - \mathbf{k}'''}} \tilde{B}_{\mathbf{k}'\delta} \tilde{B}_{\mathbf{k}'''\eta}^* \langle \tilde{V}_{\mathbf{k}''\alpha}(t') \tilde{V}_{\mathbf{k}^{\text{iv}}\beta}(t'') \rangle dt' dt'' \\ &= k^2 \hat{k}_\gamma \hat{k}_\tau (\delta_{\alpha\beta} \delta_{\gamma\delta} \delta_{\tau\eta} - \delta_{\alpha\eta} \delta_{\beta\tau} \delta_{\gamma\delta} - \delta_{\alpha\gamma} \delta_{\beta\delta} \delta_{\tau\eta} + \delta_{\alpha\gamma} \delta_{\beta\tau} \delta_{\delta\eta}) \\ &\quad \times \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} \int_0^t \int_0^t {}^0b_{\delta\eta} |\tilde{\mathbf{B}}_{\mathbf{k}'}|^2 \langle \tilde{V}_{\mathbf{k}''\alpha}(t') \tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle dt' dt''. \end{aligned} \quad (82)$$

Here, we use $\langle \tilde{V}_{\mathbf{k}''\alpha} \tilde{V}_{\mathbf{k}^{\text{iv}}\beta}^* \rangle \propto \delta_{\mathbf{k}'', \mathbf{k}^{\text{iv}}}$, see equation (53), and therefore, $\mathbf{k}^{\text{iv}} = \mathbf{k}''$ and $\mathbf{k}''' = \mathbf{k}'$. We also assume that ${}^0b_{\alpha\beta} = \text{const}$ (our first working hypothesis), and therefore, ${}^0\tilde{B}_{\mathbf{k}'\delta} {}^0\tilde{B}_{\mathbf{k}'\eta}^* = {}^0b_{\delta\eta} |\tilde{\mathbf{B}}_{\mathbf{k}'}|^2 = {}^0b_{\delta\eta} |\tilde{\mathbf{B}}_{\mathbf{k}'}|^2$.

Now, according to formula (78), we need to integrate equation (82) over all directions of the unit vector $\hat{\mathbf{k}}$. We carry out this integration in Appendix A, the result is

$$\int k^2 \langle |^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^2\hat{\mathbf{k}} = t \int_0^\infty dk' \mathcal{K}(k, k') \int k'^2 |^0\tilde{\mathbf{B}}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}}', \quad (83)$$

$$K(k, k') = k^4 \left(\frac{L}{2\pi} \right)^3 \int_0^\pi d\theta \sin^3 \theta J_{0k''} \int_0^{2\pi} d\varphi (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \left\{ \frac{k'^2 + 2k(k - k' \cos \theta) \cos^2 \varphi}{k''^2} \right. \\ \left. - \left[1 - (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \right] \frac{k^2}{k''^2} \frac{(k' - k \cos \theta)^2 + (k^2 - k'^2) \sin^2 \theta \sin^2 \varphi}{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi} \cos^2 \varphi \right\}, \quad (84)$$

$$k'' = (k^2 + k'^2 - 2kk' \cos \theta)^{1/2}, \quad (85)$$

$$\bar{\Omega}''/\Omega''_{\text{rd}} = 6 \frac{k''^2}{k_\nu^2}, \quad 2\Omega''/\Omega''_{\text{rd}} = 90 \frac{k^2}{k_\nu^2} \frac{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi}{k''^2} \sin^2 \theta \cos^2 \varphi, \quad (86)$$

where function J_{0k} is given by equation (13).

Next, we calculate the $\langle ^2\tilde{B}_{\mathbf{k}\alpha}(t) \rangle ^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.}$ term of equation (80). First, we integrate equation (32) in time, and then Fourier transform the result in space, $\mathbf{r} \rightarrow \mathbf{k}$. We have

$$^2\tilde{B}_{\mathbf{k}\eta}(t) = ik_\tau (\delta_{\eta\beta} \delta_{\tau\chi} - \delta_{\tau\beta} \delta_{\eta\chi}) \int_0^t \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} \left[^1\tilde{V}_{\mathbf{k}''\beta}(t') ^1\tilde{B}_{\mathbf{k}'\chi}(t') + ^2\tilde{V}_{\mathbf{k}''\beta}(t') ^0\tilde{B}_{\mathbf{k}'\chi} \right] dt', \quad (87)$$

where we use again the divergence free conditions, $k_\alpha ^0\tilde{B}_{\mathbf{k}\alpha} = 0$, $k_\alpha ^1\tilde{B}_{\mathbf{k}\alpha} = 0$, $k_\alpha ^1\tilde{V}_{\mathbf{k}\alpha} = 0$ and $k_\alpha ^2\tilde{V}_{\mathbf{k}\alpha} = 0$. Second, we ensemble average his equation. The second term in the brackets [...] averages out because of equation (60). Third, we multiply the averaged equation by $^0\tilde{B}_{\mathbf{k}\eta}^*$, add the complex conjugate, and use formula (81) for $^1\tilde{B}_{\mathbf{k}'\chi}$. We have

$$\langle ^2\tilde{B}_{\mathbf{k}\eta} \rangle ^0\tilde{B}_{\mathbf{k}\eta}^* + \text{c.c.} = ik_\tau (\delta_{\eta\beta} \delta_{\tau\chi} - \delta_{\tau\beta} \delta_{\eta\chi}) i(\delta_{\chi\alpha} \delta_{\gamma\delta} - \delta_{\gamma\alpha} \delta_{\chi\delta}) \\ \times \sum_{\substack{\mathbf{k}' \\ \mathbf{k}'' = \mathbf{k} - \mathbf{k}'}} \sum_{\substack{\mathbf{k}''' \\ \mathbf{k}^{\text{iv}} = \mathbf{k}' - \mathbf{k}'''}} k'_\gamma \int_0^t dt' \int_0^{t'} dt'' \langle ^1\tilde{V}_{\mathbf{k}^{\text{iv}}\alpha}(t'') ^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle ^0\tilde{B}_{\mathbf{k}\eta}^* ^0\tilde{B}_{\mathbf{k}'''\delta} + \text{c.c.} \\ = -k_\tau (\delta_{\alpha\tau} \delta_{\eta\beta} \delta_{\gamma\delta} - \delta_{\alpha\eta} \delta_{\tau\beta} \delta_{\gamma\delta} + \delta_{\delta\eta} \delta_{\gamma\alpha} \delta_{\tau\beta}) ^0\tilde{B}_{\mathbf{k}\eta}^* ^0\tilde{B}_{\mathbf{k}\delta} \\ \times \sum_{\mathbf{k}''} (k_\gamma - k''_\gamma) \int_0^t dt' \int_0^{t'} dt'' \langle ^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') ^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle + \text{c.c.} \\ = -2k_\alpha k_\beta |^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 \sum_{\mathbf{k}''} \int_0^t dt' \int_0^{t'} dt'' \langle ^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') ^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \\ + 2k_\tau (\delta_{\alpha\tau} \delta_{\eta\beta} - \delta_{\alpha\eta} \delta_{\tau\beta}) ^0b_{\eta\gamma} |^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 \sum_{\mathbf{k}''} k''_\gamma \int_0^t dt' \int_0^{t'} dt'' \langle ^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') ^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \\ = -2k^2 |^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 \left(\frac{L}{2\pi} \right)^3 \int_{-\infty}^\infty d^3\mathbf{k}'' \int_0^t dt' \int_0^{t'} dt'' \hat{k}_\alpha \langle ^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') ^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \hat{k}_\beta. \quad (88)$$

Here, we use $\langle {}^1\tilde{V}_{\mathbf{k}^{\text{iv}}\alpha} {}^1\tilde{V}_{\mathbf{k}''\beta} \rangle \propto \delta_{\mathbf{k}^{\text{iv}}, -\mathbf{k}''}$, see equation (53), and therefore, $\mathbf{k}^{\text{iv}} = -\mathbf{k}''$ and $\mathbf{k}''' = \mathbf{k}$. We use the field divergence free condition, $k_\alpha {}^0\tilde{B}_{\mathbf{k}\alpha} = 0$, this is why there are only three terms in the third line of equation (88). On the fourth line of equation (88) we use $\mathbf{k}' = \mathbf{k} - \mathbf{k}''$ to change the summation over \mathbf{k}' to the summation over \mathbf{k}'' . To obtain the fifth and sixth lines of equation (88), we use $k_\alpha {}^0\tilde{\mathbf{B}}_{\mathbf{k}\alpha} = 0$, $k_\alpha'' {}^1\tilde{V}_{-\mathbf{k}''\alpha} = 0$ and ${}^0\tilde{B}_{\mathbf{k}\eta} {}^0\tilde{B}_{\mathbf{k}\delta}^* = {}^0b_{\eta\delta} |{}^0\tilde{B}_{\mathbf{k}}|^2$ (because ${}^0b_{\eta\delta} = \text{const}$). The two terms in the sixth line of equation (88) cancel each other because of the symmetry of tensor $\langle {}^1\tilde{V}_{-\mathbf{k}''\alpha} {}^1\tilde{V}_{\mathbf{k}''\beta} \rangle$ with respect to the exchange $\alpha \leftrightarrow \beta$ [see eq. (53)]. We change the summation over \mathbf{k}'' to the integration over \mathbf{k}'' in the last line of equation (88).

Note that \mathbf{k} is perpendicular to ${}^0\hat{\mathbf{b}}$ because the field is divergence free, $k_\alpha {}^0\tilde{B}_{\mathbf{k}\alpha} = 0$. Therefore, to carry out the integration over \mathbf{k}'' in equation (88), we use a spherical system of coordinates, $d^3\mathbf{k}'' = k''^2 dk'' \sin\theta d\theta d\varphi$. Here θ is the angle between \mathbf{k}'' and \mathbf{k} , so that $\hat{\mathbf{k}}'' \cdot \hat{\mathbf{k}} = \cos\theta$, and φ is the angle between ${}^0\hat{\mathbf{b}}$ and the projection of \mathbf{k}'' on the plane perpendicular to \mathbf{k} , so that $\mu'' = \hat{\mathbf{k}}'' \cdot {}^0\hat{\mathbf{b}} = \sin\theta \cos\varphi$.¹⁸ Using equation (53), and equation (61) in the limit $t \gg \tau$, we obtain

$$\begin{aligned} & \int_0^t dt' \int_0^{t'} dt'' \hat{k}_\alpha \langle {}^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') {}^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle \hat{k}_\beta \\ &= \frac{1}{2} J_{0k''} t (1 + \bar{\Omega}''/\Omega_{\text{rd}}'')^2 \sin^2\theta \left\{ 1 - \left[1 - (1 + 2\Omega''/\Omega_{\text{rd}}'')^{-2} \right] \frac{\cos^2\theta \cos^2\varphi}{\cos^2\theta \cos^2\varphi + \sin^2\varphi} \right\}. \end{aligned} \quad (89)$$

Now, we substitute this equation and formula $d^3\mathbf{k}'' = k''^2 dk'' \sin\theta d\theta d\varphi$ into equation (88), and integrate the result over all directions of the unit vector $\hat{\mathbf{k}}$. We have

$$\begin{aligned} & \int k^2 \left[\langle {}^2\tilde{B}_{\mathbf{k}\eta} \rangle {}^0\tilde{B}_{\mathbf{k}\eta}^* + \text{c.c.} \right] d^2\hat{\mathbf{k}} = -t k^2 \int k^2 |{}^0\tilde{\mathbf{B}}_{\mathbf{k}}|^2 d^2\hat{\mathbf{k}} \left(\frac{L}{2\pi} \right)^3 \int_0^\infty k''^2 J_{0k''} dk'' \int_0^\pi \sin^3\theta d\theta \\ & \times \int_0^{2\pi} d\varphi (1 + \bar{\Omega}''/\Omega_{\text{rd}}'')^2 \left\{ 1 - \left[1 - (1 + 2\Omega''/\Omega_{\text{rd}}'')^{-2} \right] \frac{\cos^2\theta \cos^2\varphi}{\cos^2\theta \cos^2\varphi + \sin^2\varphi} \right\}. \end{aligned} \quad (90)$$

Here, $\bar{\Omega}''$, Ω'' and Ω_{rd}'' depend on k'' and on $\mu''^2 = \sin\theta \cos\varphi$, see equations (48) and (59).

Finally, we substitute equations (80), (83) and (90) into equation (78) for the magnetic energy spectrum $M(t, k)$. We choose t small enough for the quasi-linear expansion to be valid, so that $\partial_t \langle M(t, k) \rangle = [M(t, k) - M(0, k)]/t$. As a result, we obtain *the mode coupling equation* for the magnetic energy spectrum,

$$\frac{\partial M}{\partial t} = \int_0^\infty K(k, k') M(t, k') dk' - 2 \frac{\eta_T}{4\pi} k^2 M(t, k), \quad (91)$$

where the mode coupling kernel $K(k, k')$ is given by equations (84)–(86), and the turbulent diffusion constant

$$\frac{\eta_T}{4\pi} = \frac{1}{2} \left(\frac{L}{2\pi} \right)^3 \int_0^\infty k'''^2 J_{0k'''} dk''' \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\varphi (1 + \bar{\Omega}'''/\Omega_{\text{rd}}''')^2$$

¹⁸Note that these angles θ and φ have no any relation to the angles in equation (84).

$$\times \left\{ 1 - \left[1 - (1 + 2\Omega'''/\Omega_{\text{rd}}''')^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}, \quad (92)$$

$$\bar{\Omega}'''/\Omega_{\text{rd}}''' = 6 \frac{k'''^2}{k_\nu^2}, \quad 2\Omega'''/\Omega_{\text{rd}}''' = 90 \frac{k'''^2}{k_\nu^2} \sin^2 \theta \cos^2 \varphi (1 - \sin^2 \theta \cos^2 \varphi). \quad (93)$$

The function J_{0k} is given by equation (13). Here, we use equations (48) and (59). To obtain equation (86) we use equations (A8). To obtain equation (93), we use formula $\mu''^2 = \sin \theta \cos \varphi$. In the last three equations for the turbulent diffusion constant we replace k'' by k''' in order to distinguish it from k'' in the equations (84) for the coupling kernel. Note, that according to the mode coupling equation (91), the k'' modes of the turbulence interact with the k' modes of the magnetic field to change the energy in the k modes of the magnetic field.

In the kinematic turbulent dynamo case, we take the limit $\Omega_{\text{rd}} \rightarrow \infty$. In this limit, as one might expect, after integrating over φ , equation (84) reduces to the mode coupling equation of Kulsrud and Anderson (1992). Equation (92) for the turbulent diffusion constant, after integrating over both θ and φ , reduces to the corresponding equation of Kulsrud and Anderson (1992) [see also Kraichnan and Nagarajan (1967), who obtained the same equation in a different form].

4.3. The Magnetic Energy Spectrum on Subviscous Scales

Equation (91), which gives the evolution of the magnetic energy spectrum in the magnetized turbulent dynamo theory, is the principal result of this paper. However, this equation is rather complicated for easy interpretation. In this section we limit ourselves to the evolution of the magnetic energy spectrum on small subviscous scales, which is of great interest during the magnetized dynamo stage in a protogalaxy¹⁹. In this limit, $k \gg k_\nu$, and the integro-differential equation (91) for the magnetic spectrum simplifies to an ordinary differential equation.

Let us refer to equation (84) for the mode coupling kernel $K(k, k')$. Function $J_{0k''}$ cuts off at the viscous wave number k_ν [see eq. (13)]. Therefore, in the large- k limit, $k \gg k_\nu$, we have $k'' \sim |k - k'| \ll k, k'$, and can expand the kernel $K(k, k')$. However, the simplest way of calculations is to introduce an arbitrary function of k , $F(k)$, which varies slowly in the region $k \gg k_\nu$, and vanishes outside of this region (Kulsrud & Anderson 1992). To derive the mode coupling equation on small (subviscous) scales, we calculate the following integral

$$\begin{aligned} \int_0^\infty F(k) \frac{\partial M}{\partial t} dk &= \frac{1}{4\pi\rho} \left(\frac{L}{2\pi} \right)^3 \int_{-\infty}^\infty F \frac{\partial \langle |\tilde{\mathbf{B}}_{\mathbf{k}}|^2 \rangle}{\partial t} d^3\mathbf{k} = \frac{1}{4\pi\rho} \left(\frac{L}{2\pi} \right)^3 \int_{-\infty}^\infty F \frac{\langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle - |\tilde{\mathbf{B}}_{\mathbf{k}}(0)|^2}{t} d^3\mathbf{k} \\ &= \frac{1}{4\pi\rho} \left(\frac{L}{2\pi} \right)^3 \frac{1}{t} \left\{ \int_{-\infty}^\infty F(k) \langle |\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^3\mathbf{k} + \int_{-\infty}^\infty F(k) \left[\langle \tilde{B}_{\mathbf{k}\alpha}(t) \rangle^0 \tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.} \right] d^3\mathbf{k} \right\} \end{aligned}$$

¹⁹The time evolution of the magnetic energy on large scales is slow because it is set by the turnover time of large turbulent eddies. As a result, the large-scale field is not of significant interest until the small-scale field saturates, the magnetized dynamo stage ends and the inverse cascade stage begins.

$$= \frac{\Gamma}{5} \int_0^\infty F(k) \left[k^2 \frac{\partial^2 M}{\partial k^2} - (\Lambda_1 - 1)k \frac{\partial M}{\partial k} + \Lambda_0 M \right] dk. \quad (94)$$

Here, to obtain the first line, we use equation (78) and replace the time derivative by the time finite difference, assuming that t is small, and our quasi-linear expansion is valid. To obtain the second line, we use equation (80). The derivation of the final result [the 3rd line] is given in Appendix B. The constants Γ , Λ_1 and Λ_0 are

$$\begin{aligned} \Gamma &= \frac{5}{2} \left(\frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} dk \int_0^\pi d\theta \sin^3 \theta \cos^2 \theta \\ &\quad \times \int_0^{2\pi} d\varphi (1 + \bar{\Omega}/\Omega_{\text{rd}})^2 \left\{ 1 - \left[1 - (1 + 2\Omega/\Omega_{\text{rd}})^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}, \end{aligned} \quad (95)$$

$$\begin{aligned} \Lambda_1 &= -3 + \frac{5}{\Gamma} \left(\frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} dk \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi (1 + \bar{\Omega}/\Omega_{\text{rd}})^2 \left\{ 2 \cos^2 \theta \cos^2 \varphi + \frac{1}{2} \sin^2 \theta \right. \\ &\quad \left. - \left[1 - (1 + 2\Omega/\Omega_{\text{rd}})^{-2} \right] \left(2 + \frac{1}{2} \frac{\sin^2 \theta}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right) \cos^2 \theta \cos^2 \varphi \right\}, \end{aligned} \quad (96)$$

$$\begin{aligned} \Lambda_0 &= 2 + \frac{5}{\Gamma} \left(\frac{L}{2\pi} \right)^3 \int_0^\infty k^4 J_{0k} dk \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi (1 + \bar{\Omega}/\Omega_{\text{rd}})^2 \left\{ 2 \sin^2 \theta \cos^2 \varphi - \frac{1}{2} \sin^2 \theta \right. \\ &\quad \left. + \left[1 - (1 + 2\Omega/\Omega_{\text{rd}})^{-2} \right] \left(\cos^2 \theta - \sin^2 \theta + \frac{1}{2} \frac{\sin^2 \theta \cos^2 \theta}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right) \cos^2 \varphi \right\}, \end{aligned} \quad (97)$$

$$\bar{\Omega}/\Omega_{\text{rd}} = 6 \frac{k^2}{k_\nu^2}, \quad 2\Omega/\Omega_{\text{rd}} = 90 \frac{k^2}{k_\nu^2} \sin^2 \theta \cos^2 \varphi (1 - \sin^2 \theta \cos^2 \varphi), \quad (98)$$

and function J_{0k} is given by equation (13).

Equation (94) is valid for an arbitrary function F . As a result, the integrands in the left- and right-hand-side of this equation should be equal, and we finally obtain the mode-coupling equation for the magnetic energy spectrum $M(t, k)$ on small (subviscous) scales

$$\frac{\partial M}{\partial t} = \frac{\Gamma}{5} \left[k^2 \frac{\partial^2 M}{\partial k^2} - (\Lambda_1 - 1)k \frac{\partial M}{\partial k} + \Lambda_0 M \right]. \quad (99)$$

The constant Γ and the constant dimensionless numbers Λ_0 and Λ_1 can easily be calculated numerically. The result is (Malyshkin 2001)

$$\Gamma \approx 11 (U_0 L/\nu)^{1/2} (U_0/L), \quad \Lambda_1 \approx 2, \quad \Lambda_0 \approx 5, \quad (100)$$

and therefore,

$$\frac{\partial M}{\partial t} = 2.2 \left(\frac{U_0 L}{\nu} \right)^{1/2} \frac{U_0}{L} \left[k^2 \frac{\partial^2 M}{\partial k^2} - k \frac{\partial M}{\partial k} + 5M \right]. \quad (101)$$

(It is easy to see that $\Lambda_1 = 2$ and $\Lambda_0 = 5$ correspond to the limit $\Omega_{\text{rd}} \ll \nu k_\nu^2$.)

If for the moment we consider the kinematic turbulent dynamo, then we take the limit $\Omega_{\text{rd}} \rightarrow \infty$ in equations (95)–(97). After integrating over φ and θ , we have $\Gamma = \gamma_o$, where γ_o is the Kulsrud-Anderson (1992) magnetic energy growth rate, $\Lambda_1 = 3$ and $\Lambda_0 = 6$. In this case equation (99)

coincides with the corresponding equation of Kulsrud and Anderson (1992) and with the corresponding equation of Kazantsev (1968), as one might expect (note that Kazantsev’s equation is given in the Fourier space).

Now, assume that $M(t, k_{\text{ref}})$ is known as a function of time at some reference wave number $k = k_{\text{ref}}$. Then the solution of (99) is

$$M(t, k) = \int_{-\infty}^t M(t', k_{\text{ref}}) G(k/k_{\text{ref}}, t - t') dt', \quad (102)$$

where the Green’s function $G(k, t)$ is

$$G(k, t) = \sqrt{\frac{5}{4\pi}} \frac{k^{\Lambda_1/2} \ln k}{\Gamma^{1/2} t^{3/2}} e^{(\Gamma/5)(\Lambda_0 - \Lambda_1^2/4)t} e^{-5 \ln^2 k / 4\Gamma t} = \sqrt{\frac{5}{4\pi}} \frac{k \ln k}{\Gamma^{1/2} t^{3/2}} e^{(4\Gamma/5)t} e^{-5 \ln^2 k / 4\Gamma t}. \quad (103)$$

[This Green function can be obtained by applying the Laplace transformation in time, $t \rightarrow s$, to equation (99), see Kulsrud & Anderson 1992; Malyshkin 2001]. To obtain the final result in equation (103), we use equations (100) for Λ_1 and Λ_0 .

According to equations (102) and (103), we see that a “signal” $M(t, k_{\text{ref}})$, at zero time, will increase exponentially as $e^{(4/5)\Gamma t}$ and will extend down to the scale $k_{\text{peak}} \approx e^{(4/5)\Gamma t} k_{\text{ref}}$, where k_{peak} is the peak of function $kG(k, t)$, (of course, the field scale can not become less than the resistivity scale). As a result, in the magnetized dynamo theory the magnetic energy tends to quickly propagate to very small subviscous scales, the same way as it does in the kinematic dynamo theory (Kulsrud & Anderson 1992; Schekochihin, Boldyrev, & Kulsrud 2002). However, the tail of the magnetic energy spectrum on $k_{\text{ref}} \lesssim k \lesssim k_{\text{peak}}$ scales increases with the wave number as $\propto k^{\Lambda_1/2} = k$ instead of $\propto k^{3/2}$ in the kinematic theory (Kulsrud and Anderson 1992). Note, that according to equations (75) and (100), the growth rate of the Green’s function, $(4/5)\Gamma$, is approximately equal to a half of the growth rate of the total magnetic energy (the latter one is 2γ).

5. Discussion and Conclusions

In this paper we have developed a theoretical basis for the magnetized turbulent dynamo, which operates in protogalaxies, where the plasma is fully ionized, and the viscosity is the Braginskii tensor viscosity. The results of the kinematic dynamo theory, already seem to support the primordial (protogalactic) dynamo origin of cosmic magnetic fields (Kulsrud *et al.* 1997). The results that we have obtained for the magnetized dynamo, further support this primordial origin theory. This is because the number of the magnetic energy e-foldings during the magnetized turbulent dynamo stage in a protogalaxy, given by equation (76), is up to ten time larger than that in the kinematic dynamo theory. This number of e-foldings is more than large enough for the magnetic field in the protogalaxy to grow from its seed value, provided by the Biermann battery, up to the field-turbulence energy equipartition value. The number of e-foldings of the magnetic energy on the

viscous scale, which is determined by the growth rate $(4/5)\Gamma$ of the Green's function (103), is less by one half, but it is still sufficiently large ²⁰.

Another our prediction is that the tail of the magnetic energy spectrum on small subviscous scales increases with the wave number as $\propto k$ [see the Green's function (103)], instead of $\propto k^{3/2}$ in the kinematic theory ²¹. Therefore, in the magnetized dynamo the magnetic energy is slightly shifted to larger scales as compared to the kinematic dynamo case.

The Green's function solution (103) indicates that in the magnetized dynamo theory the magnetic energy tends to quickly propagate to very small subviscous scales, similar to the kinematic dynamo case. On the other hand, the observed cosmic fields have rather large correlation lengths. Therefore, the magnetic field lines must be unwrapped on small scales by the Lorentz tension forces, while the field energy is transferred and amplified on larger scales during the inverse cascade stage. This most important and most interesting stage happens when the magnetic field energy is comparable to the turbulent kinetic energy. In the final part of this paper let us discuss the significance of the Braginskii viscosity for the inverse cascade in more details.

First, note that the theory of the inverse cascade in a plasma with the regular isotropic (non-Braginskii) viscosity has a difficulty of unwrapping of the small-scale magnetic field lines. This difficulty can be understood as follows ²². The equation for the turbulent velocities \mathbf{V} in the plasma with the isotropic viscosity, including the Lorentz forces, is (Landau & Lifshitz 1984)

$$\partial_t V_\alpha = -P_{,\alpha} + f_\alpha + \nu \Delta V_\alpha + (1/4\pi\rho)(\mathbf{B} \cdot \nabla)B_\alpha - (\mathbf{V} \cdot \nabla)V_\alpha \quad (104)$$

[compare with eq. (4)]. We can estimate the unwrapping velocity, V_{unwrap} , by Fourier transforming equation (104) in space, $\mathbf{r} \rightarrow \mathbf{k}$, and then balancing the viscous and the inertial forces against the magnetic tension force. The isotropic viscosity dominates on small scales, and we have

$$\nu k^2 V_{\text{unwrap}} \sim (1/4\pi\rho)k_{\parallel} B^2, \quad V_{\text{unwrap}} \sim (k_{\parallel}/k) (k_{\nu}/k) (V_A^2/\nu k_{\nu}) \ll V_A, \quad (105)$$

where V_A is the Alfven speed. At the time of the field-turbulence energy equipartition on the viscous scale, the Alfven speed is $V_A \sim \nu k_{\nu}$, and before this equipartition $V_A < \nu k_{\nu}$. Since in the kinematic dynamo theory the field lines have a folding pattern, $k \gg k_{\parallel} \sim k_{\nu}$ (see Fig. 2), the unwrapping velocity (105) is small compared to the Alfven speed, even at the equipartition time. In other words, since $V_A \sim V$ at the energy equipartition, then $k_{\parallel} V_A \sim \gamma$ (here 2γ is the magnetic energy growth rate), and the unwrapping rate, $k_{\parallel} V_{\text{unwrap}}$, is much smaller than γ . This means that the field continues to grow on the viscous and subviscous scales even beyond the energy equipartition.

²⁰Of course, our results (71)–(77) for the magnetic energy growth rate are sensitive to the value of the physical parameter Ω_{rd} , which is estimated in equation (59). We also left out the finite time correlation effects. Therefore, our result for the number of magnetic energy e-foldings should be viewed as an estimate, valid within a factor of order two. However, it is important that the number of e-foldings, which we found, is a very large number.

²¹This our prediction is not sensitive to the value of Ω_{rd} , as long as $\Omega_{\text{rd}} \ll \nu k_{\nu}^2$.

²²Cowley, Kulsrud, & Schekochihin 2001, private communications.

However, in the case of the magnetized turbulent dynamo the viscosity term in equation (104) is modified, and the anti-unwrapping argument does not apply. Indeed, the field unwrapping velocity is parallel and varies in the direction perpendicular to the magnetic field lines. The large velocity gradient perpendicular to the field lines, which leads to a large perpendicular stress in the isotropic viscosity case, is suppressed in the case of the Braginskii viscous forces (because the transfer of the ion momentum in the perpendicular direction is inhibited). Therefore, in the magnetized dynamo theory $k_{\parallel} V_{\text{unwrap}} \sim k_{\parallel} V_A \sim \gamma$ at the equipartition, and the magnetic field strength saturates on the viscous and subviscous scales. As a result, the Braginskii viscosity makes the inverse cascade of the magnetic energy more likely, because the larger turbulent eddies do not need to deliver their energy to the field on the viscous and subviscous scales (Kulsrud 2000).

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A. Details of the derivation of the mode coupling equation

Equation (82) has four terms in the right-hand-side. Therefore, we have

$$\int k^2 \langle |{}^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^2\hat{\mathbf{k}} = \mathcal{T} - \mathcal{T}' - \mathcal{T}'' + \mathcal{T}''', \quad (\text{A1})$$

where there are also four terms,

$$\mathcal{T} = k^4 \left(\frac{L}{2\pi} \right)^3 \int_0^\infty dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}}' \int \mu^2 d^2\hat{\mathbf{k}} \int_0^t \int_0^t \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'', \quad (\text{A2})$$

$$\mathcal{T}' = k^4 \left(\frac{L}{2\pi} \right)^3 \int_0^\infty dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}}' \int \mu d^2\hat{\mathbf{k}} \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle \hat{k}_\beta dt' dt'', \quad (\text{A3})$$

$$\mathcal{T}'' = \mathcal{T}'^* = \mathcal{T}', \quad (\text{A4})$$

$$\mathcal{T}''' = k^4 \left(\frac{L}{2\pi} \right)^3 \int_0^\infty dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}}' \int d^2\hat{\mathbf{k}} \int_0^t \int_0^t \hat{k}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle \hat{k}_\beta dt' dt'', \quad (\text{A5})$$

Here, $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$, $\mu = (\hat{\mathbf{k}} \cdot {}^0\hat{\mathbf{b}})$ [see eq. (44)], and we replace the summation over \mathbf{k}' by the double integration over k' and $\hat{\mathbf{k}}'$. Next, refer to Figure 3A. Vector \mathbf{k}' is perpendicular to ${}^0\hat{\mathbf{b}}$ because the field is divergence free, $k'_\alpha {}^0\tilde{B}_{\mathbf{k}'\alpha} = 0$. The following useful equations are valid (Malyskhin 2001),

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' = \cos \theta, \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}'' = (k - k' \cos \theta)/k'', \quad k'' = (k^2 + k'^2 - 2kk' \cos \theta)^{1/2}, \quad (\text{A6})$$

$$\mu = \hat{\mathbf{k}} \cdot {}^0\hat{\mathbf{b}} = \sin \theta \cos \varphi, \quad d^2\hat{\mathbf{k}} = \sin \theta d\theta d\varphi, \quad (\text{A7})$$

$$\mu'' = {}^0\hat{\mathbf{b}} \cdot \hat{\mathbf{k}}'' = \frac{k}{k''} \sin \theta \cos \varphi, \quad 1 - \mu''^2 = \frac{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi}{k''^2}. \quad (\text{A8})$$

Using these formulas, equation (53) and equation (62) in the limit $t \gg \tau$, it is straightforward to calculate the double time integral terms in equations (A2)–(A5),

$$\int_0^t \int_0^t \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'' = J_{0k''} t (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \left[1 + (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \right], \quad (\text{A9})$$

$$\int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle \hat{k}_\beta dt' dt'' = J_{0k''} t \left(\frac{1 + \bar{\Omega}''/\Omega''_{\text{rd}}}{1 + 2\Omega''/\Omega''_{\text{rd}}} \right)^2 \frac{k'(k' - k \cos \theta) \sin \theta \cos \varphi}{k''^2}, \quad (\text{A10})$$

$$\begin{aligned} \int_0^t \int_0^t \hat{k}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle \hat{k}_\beta dt' dt'' &= J_{0k''} t (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \frac{k'^2 \sin^2 \theta}{(k' - k \cos \theta)^2 + k^2 \sin^2 \theta \sin^2 \varphi} \\ &\times \left[\sin^2 \varphi + (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \frac{(k' - k \cos \theta)^2 \cos^2 \varphi}{k''^2} \right]. \quad (\text{A11}) \end{aligned}$$

Here, $\bar{\Omega}''$, Ω'' and Ω''_{rd} depend on k'' and on μ''^2 [see eqs. (48), (59)]. In turn, k'' and μ'' are functions of k , k' , θ and φ [see eqs. (A6), (A8)]. Now, we substitute formulas (A7) and (A9)–(A11) into equations (A2)–(A5). The factors, which we obtain in these equations after integration over $d^2\hat{\mathbf{k}} = \sin \theta d\theta d\varphi$, depend only on k and k' , so they can be exchanged with the integrations over $d^2\hat{\mathbf{k}}'$. Combining the results together in equation (A1), we finally obtain equations (83)–(86).

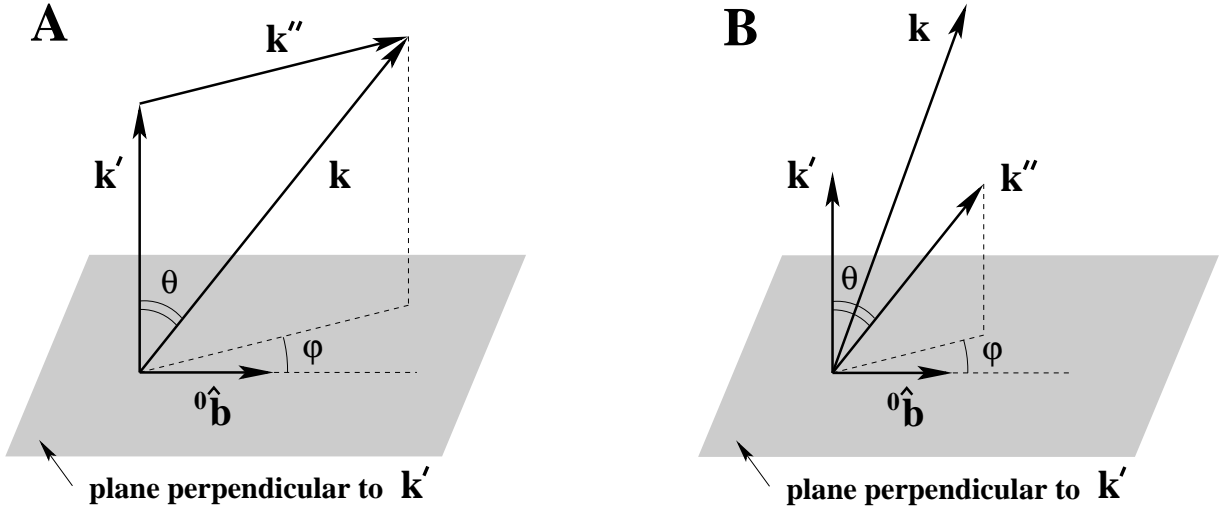


Fig. 3.— Relative position of vectors $\mathbf{\hat{b}}$, \mathbf{k}' , \mathbf{k}'' and $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$ in space. Note, that \mathbf{k}' is perpendicular to $\mathbf{\hat{b}}$ because the magnetic field is divergence free. The plot on the left (A) refers to Appendix A, the plot on the right (B) refers to Appendix B.

B. The derivation of equation (94)

First, we use formula $d^3\mathbf{k} = dk d^2\hat{\mathbf{k}}$, equations (A1)–(A5) and equation (88) to obtain the term in the brackets $\{\dots\}$ in equation (94),

$$\begin{aligned} \{\dots\} &= \int_0^\infty F(k) dk \int k^2 \langle |^1\tilde{\mathbf{B}}_{\mathbf{k}}(t)|^2 \rangle d^2\hat{\mathbf{k}} + \int_0^\infty F(k) dk \int k^2 \left[\langle ^2\tilde{B}_{\mathbf{k}\alpha}(t) \rangle ^0\tilde{B}_{\mathbf{k}\alpha}^* + \text{c.c.} \right] d^2\hat{\mathbf{k}} \\ &= \left(\frac{L}{2\pi} \right)^3 \left[\mathcal{T}_\diamond - \mathcal{T}'_\diamond - \mathcal{T}''_\diamond + \mathcal{T}'''_\diamond + \mathcal{T}^{\text{iv}}_\diamond \right], \end{aligned} \quad (\text{B1})$$

$$\begin{aligned} \mathcal{T}_\diamond &= \int_{-\infty}^\infty \int_{-\infty}^\infty \mu^2 k^2 F(k) |^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' d^3\mathbf{k} \int_0^t \int_0^t \langle ^1\tilde{V}_{\mathbf{k}''\alpha}(t') ^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'' \\ &= \int_{-\infty}^\infty |^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^\infty \mu''^2 k''^2 F(k) d^3\mathbf{k}'' \int_0^t \int_0^t \langle ^1\tilde{V}_{\mathbf{k}''\alpha}(t') ^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'', \end{aligned} \quad (\text{B2})$$

$$\begin{aligned} \mathcal{T}'_\diamond &= \int_{-\infty}^\infty \int_{-\infty}^\infty \mu k F(k) |^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' d^3\mathbf{k} \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle ^1\tilde{V}_{\mathbf{k}''\alpha}(t') ^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k_\beta dt' dt'' \\ &= \int_{-\infty}^\infty |^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^\infty \mu'' k'' F(k) d^3\mathbf{k}'' \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle ^1\tilde{V}_{\mathbf{k}''\alpha}(t') ^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'', \end{aligned} \quad (\text{B3})$$

$$\mathcal{T}''_\diamond = \mathcal{T}'_\diamond, \quad (\text{B4})$$

$$\begin{aligned} \mathcal{T}'''_\diamond &= \int_{-\infty}^\infty \int_{-\infty}^\infty F(k) |^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' d^3\mathbf{k} \int_0^t \int_0^t k_\alpha \langle ^1\tilde{V}_{\mathbf{k}''\alpha}(t') ^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k_\beta dt' dt'' \\ &= \int_{-\infty}^\infty |^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^\infty F(k) d^3\mathbf{k}'' \int_0^t \int_0^t k'_\alpha \langle ^1\tilde{V}_{\mathbf{k}''\alpha}(t') ^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'', \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} \mathcal{T}^{\text{iv}}_\diamond &= -2 \int_{-\infty}^\infty F(k) |^0\tilde{B}_{\mathbf{k}}|^2 d^3\mathbf{k} \int_{-\infty}^\infty d^3\mathbf{k}'' \int_0^t dt' \int_0^{t'} dt'' k_\alpha \langle ^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') ^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle k_\beta \\ &= - \int_{-\infty}^\infty F(k) |^0\tilde{B}_{\mathbf{k}}|^2 d^3\mathbf{k} \int_{-\infty}^\infty d^3\mathbf{k}'' \int_0^t \int_0^t k_\alpha \langle ^1\tilde{V}_{-\mathbf{k}''\alpha}(t'') ^1\tilde{V}_{\mathbf{k}''\beta}(t') \rangle k_\beta dt' dt''. \end{aligned} \quad (\text{B6})$$

Here, to obtain the final results in equations (B2)–(B5), we change the integration over \mathbf{k} in these equations to integration over \mathbf{k}'' , using $\mathbf{k} = \mathbf{k}' + \mathbf{k}''$. We use $k_\alpha^{\prime 1} \tilde{V}_{\mathbf{k}''\alpha}$, and therefore, $k_\alpha^1 \tilde{V}_{\mathbf{k}''\alpha} = k'_\alpha {}^1\tilde{V}_{\mathbf{k}''\alpha}$. We also use $\mu k = {}^0\hat{\mathbf{b}} \cdot \mathbf{k} = {}^0\hat{\mathbf{b}} \cdot \mathbf{k}'' = k'' ({}^0\hat{\mathbf{b}} \cdot \hat{\mathbf{k}}'') = \mu'' k'' = k'' \sin \theta \cos \varphi$ (see Fig. 3B). To obtain the final result in equation (B6), we use equations (53) and (62). Next, we calculate \mathcal{T}_\diamond , \mathcal{T}'_\diamond and \mathcal{T}'''_\diamond separately, up to the second order in $k'' \ll k, k'$. Refer to Figure 3B.

First, following Kulsrud and Anderson (1992), in equations (B2)–(B5) we expand $F(k)$ in $k'' \ll k$ at point k' up to the second order. We have, see Figure 3B,

$$k = k' + k'' \cos \theta + (k''^2/2k') \sin^2 \theta, \quad (\text{B7})$$

$$F(k) = F(k') + \frac{dF}{dk'} k'' \cos \theta + \frac{1}{2k'} \frac{dF}{dk'} k''^2 \sin^2 \theta + \frac{1}{2} \frac{d^2F}{dk'^2} k''^2 \cos^2 \theta, \quad (\text{B8})$$

(Kulsrud & Anderson 1992; Malyshkin 2001).

Second, we calculate \mathcal{T}_\diamond , given by equation (B2). Because $\mu''^2 k''^2$ is of the second order in k'' ,

we need to keep only the zero order term in expansion (B8) for $F(k)$. Thus, we have

$$\begin{aligned}\mathcal{T}_\diamond &= \int_{-\infty}^{\infty} F(k') |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} \mu''^2 k''^2 d^3\mathbf{k}'' \int_0^t \int_0^t \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\alpha}^*(t'') \rangle dt' dt'' \\ &= t \int_0^\infty F(k') dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}}' \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \\ &\quad \times \int_0^{2\pi} d\varphi \cos^2 \varphi (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \left[1 + (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \right],\end{aligned}\tag{B9}$$

where we use $d^3\mathbf{k}'' = k''^2 dk'' \sin \theta d\theta d\varphi$, $\mu'' = \sin \theta \cos \varphi$ and equation (A9). Here and below, functions $\bar{\Omega}''$, Ω'' and Ω''_{rd} depend on k'' and $\mu''^2 = \sin^2 \theta \cos^2 \varphi$, see equations (48) and (59).

Third, we calculate \mathcal{T}'_\diamond , given by equation (B3). Because $\mu'' k''$ is of the first order in k'' , we need to keep only the zero and the first order terms in expansion (B8) for $F(k)$. We have

$$\begin{aligned}\mathcal{T}'_\diamond &= \int_{-\infty}^{\infty} F(k') |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} \mu'' k'' d^3\mathbf{k}'' \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' \\ &\quad + \int_{-\infty}^{\infty} \frac{dF}{dk'} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} \mu'' k''^2 \cos \theta d^3\mathbf{k}'' \int_0^t \int_0^t {}^0\hat{b}_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' \\ &= -t \int_0^\infty k' \frac{dF}{dk'} dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}}' \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \cos^2 \theta \\ &\quad \times \int_0^{2\pi} d\varphi \cos^2 \varphi (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2}.\end{aligned}\tag{B10}$$

Here, we use equations (53), (62), $k'_\alpha {}^0\hat{b}_\alpha = 0$, $d^3\mathbf{k}'' = k''^2 dk'' \sin \theta d\theta d\varphi$, $\mu'' = \sin \theta \cos \varphi$. The first term in the first line of equation (B10) vanishes after the integration over θ because the integrand is an odd function of $\cos \theta$.

Fourth, we calculate \mathcal{T}'''_\diamond , given by equation (B5), keeping all terms in expansion (B8) for $F(k)$,

$$\begin{aligned}\mathcal{T}'''_\diamond &= \int_{-\infty}^{\infty} F(k') |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} d^3\mathbf{k}'' \int_0^t \int_0^t k'_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' \\ &\quad + \int_{-\infty}^{\infty} \frac{dF}{dk'} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} k'' \cos \theta d^3\mathbf{k}'' \int_0^t \int_0^t k'_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2k'} \frac{dF}{dk'} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} k''^2 \sin^2 \theta d^3\mathbf{k}'' \int_0^t \int_0^t k'_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' \\ &\quad + \int_{-\infty}^{\infty} \frac{1}{2} \frac{d^2 F}{dk'^2} |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^3\mathbf{k}' \int_{-\infty}^{\infty} k''^2 \cos^2 \theta d^3\mathbf{k}'' \int_0^t \int_0^t k'_\alpha \langle {}^1\tilde{V}_{\mathbf{k}''\alpha}(t') {}^1\tilde{V}_{\mathbf{k}''\beta}^*(t'') \rangle k'_\beta dt' dt'' \\ &= -\mathcal{T}^{\text{iv}}_\diamond \\ &\quad + \frac{t}{2} \int_0^\infty k' \frac{dF}{dk'} dk' \int k'^2 |{}^0\tilde{B}_{\mathbf{k}'}|^2 d^2\hat{\mathbf{k}}' \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^5 \theta \\ &\quad \times \int_0^{2\pi} d\varphi (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \left\{ 1 - \left[1 - (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}\end{aligned}$$

$$\begin{aligned}
& + \frac{t}{2} \int_0^\infty k'^2 \frac{d^2 F}{dk'^2} dk' \int k'^2 |\tilde{B}_{\mathbf{k}'}|^2 d^2 \hat{\mathbf{k}}' \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \cos^2 \theta \\
& \times \int_0^{2\pi} d\varphi (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \left\{ 1 - \left[1 - (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}. \quad (\text{B11})
\end{aligned}$$

Here, the first term in the first line is equal to minus $\mathcal{T}_\diamond^{\text{iv}}$, given by equation (B6). To calculate the other three terms (on the 2nd, 3rd and 4th lines), we again use equations (53), (62), $k'_\alpha \hat{b}_\alpha = 0$, $d^3 \mathbf{k}'' = k''^2 dk'' \sin \theta d\theta d\varphi$, $\mu'' = \sin \theta \cos \varphi$. The term in the second line of equation (B11) vanishes after the integration over θ because the integrand is an odd function of $\cos \theta$.

Next, we substitute equations (B4), (B9), (B10) and (B11) into equation (B1). The $-\mathcal{T}_\diamond^{\text{iv}}$ term of equation (B11) cancels the $\mathcal{T}_\diamond^{\text{iv}}$ term in equation (B1). Then, we substitute the result, obtained in equation (B1), into the second line of equation (94) and, using equation (78), we find

$$\int_0^\infty F(k) \frac{\partial M}{\partial t} dk = \left(\frac{L}{2\pi} \right)^3 \int_0^\infty \left[\lambda_0 F(k') + \lambda_1 k' \frac{dF}{dk'} + \lambda_2 k'^2 \frac{d^2 F}{dk'^2} \right] M(0, k') dk', \quad (\text{B12})$$

$$\lambda_0 = \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi \cos^2 \varphi (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \left[1 + (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \right], \quad (\text{B13})$$

$$\begin{aligned}
\lambda_1 = & \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \int_0^{2\pi} d\varphi (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \left\{ 2 \cos^2 \theta \cos^2 \varphi + \frac{1}{2} \sin^2 \theta \right. \\
& \left. - \left[1 - (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \right] \left(2 + \frac{1}{2} \frac{\sin^2 \theta}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right) \cos^2 \theta \cos^2 \varphi \right\}, \quad (\text{B14})
\end{aligned}$$

$$\begin{aligned}
\lambda_2 = & \frac{1}{2} \int_0^\infty k''^4 J_{0k''} dk'' \int_0^\pi d\theta \sin^3 \theta \cos^2 \theta \\
& \times \int_0^{2\pi} d\varphi (1 + \bar{\Omega}''/\Omega''_{\text{rd}})^2 \left\{ 1 - \left[1 - (1 + 2\Omega''/\Omega''_{\text{rd}})^{-2} \right] \frac{\cos^2 \theta \cos^2 \varphi}{\cos^2 \theta \cos^2 \varphi + \sin^2 \varphi} \right\}. \quad (\text{B15})
\end{aligned}$$

Now, we integrate the right-hand-side of equation (B12) by parts over some extent in k' and choose $F(k')$, so that it and its derivative dF/dk' vanish at the end points. We finally obtain the third line of equation (94), by dropping the double primes, $''$, in equations (B13)–(B15), and introducing the new constants Γ , Λ_1 and Λ_0 , given by equations (95)–(97).

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